AN ELEMENTARY GUIDE TO THE ADAMS-NOVIKOV EXT

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The Adams-Novikov spectral sequence for the Brown-Peterson spectrum

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*) \implies \pi^S_{s-t}(S^0)_{(p)}$$

has been one of the most successful tools in the understanding of stable homotopy groups of spheres. However, already the calculation of the algebraic E_2-page presents some prohibitive difficulties and led to the development of computational tools such as the chromatic spectral sequence of [2].

The purpose of this exposition is to introduce the reader to the Adams-Novikov Ext without using any of the heavy machinery. In particular, we will calculate the 1-line of the Adams-Novikov spectral sequence, that is $\mathrm{Ext}^{1,*}(BP_*)$. This was originally the result of Novikov [3], who also related Ext^1 with the image of the J-homomorphism and Miller-Ravenel-Wilson [2], who used the chromatic spectral sequence to obtain a lot more general results. We will then move on to other examples, including the image of Ext^1 in the classical Adams spectral sequence and a short detour of the higher Greek letter elements.

1. Hopf algebroids and BP_*

We assume the reader is familiar with the general theory of Hopf algebroids and their homological algebra and with the construction of the Adams-Novikov spectral sequence. The reference for these is [5, A1.1, A1.2, 2.2]. We will briefly recall the relevant algebraic notions.

A *Hopf algebroid* over a ground ring K is a pair (A, Γ) of commutative K-algebras equipped with the left and right units $\eta_L, \eta_R : A \to \Gamma$, counit $\Gamma \to A$, conjugation $\Gamma \to \Gamma$ and a coproduct

$$\Delta : \Gamma \to \Gamma \otimes_A \Gamma$$

all of which are K-algebra homomorphisms satisfying a number of compatibility axioms. In the definition of Δ we treat Γ as a left A-module via η_L and as a right A-module via η_R. An example of a Hopf algebroid is any Hopf algebra, in particular $(\mathbb{F}_p, \mathcal{A}_p^*)$ where $\mathcal{A}_p^*$ denotes the dual mod p Steenrod algebra.

A *left* Γ*-comodule* is a left A-module M equipped with a map $M \to \Gamma \otimes_A M$, which again satisfies the usual compatibility axioms. Right comodules are defined analogously. Note that A itself is a left Γ-comodule via the map $a \mapsto 1 \otimes a$, which can be identified with the right unit $\eta_R : A \to \Gamma = \Gamma \otimes_A A$. Similarly, A is a right Γ-comodule via the map $a \mapsto a \otimes 1$ which can be identified with η_L.

For a right Γ-comodule M and a left Γ-comodule N we define the *cotensor product* $M \square N$ as

$$M \square N = \ker(\psi_M \otimes \mathrm{id}_N - id_M \otimes \psi_N : M \otimes_A N \to M \otimes_A \Gamma \otimes_A N).$$

In particular, if $M = A$ with its right module structure then $A \square N$ is the submodule of primitive elements of N (i.e. $n \in N$ such that $\psi_N(n) = 1 \otimes n$). For every M the functor

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$\mathrm{Cotor}^i_\Gamma(M,N)$ is the i-th right derived functor of $N \mapsto M\Box_\Gamma N$. When $M = A$ we abbreviate notation to $\mathrm{Cotor}^i(N)$. In due course this will always be denoted by $\mathrm{Ext}^i(N)$, due to the isomorphism between $\mathrm{Hom}_\Gamma(A,-)$ and $A\Box_\Gamma-$, as explained in [5, p.310].

For the calculation of $\mathrm{Ext}(N)$ it is customary to treat the left Γ-comodule N as a right Γ-comodule via a device which employs the conjugation map of the algebroid. The standard left and right Γ-comodule structures on A correspond to each other under this operation. Now it follows (see [5, A1.2.12]) that $\mathrm{Ext}^i(N)$ is the i-th cohomology group of the cobar complex $\{C^s(N)\}_{s\geq 0}$, where $C^s(N) = N \otimes_A \Gamma^{\otimes_A s}$:

$$N \xrightarrow{d} N \otimes_A \Gamma \xrightarrow{d} N \otimes_A \Gamma \otimes_A \Gamma \longrightarrow \cdots \tag{1}$$

We write a typical element of $C^s(N)$ as $n\gamma_1|\cdots|\gamma_s$ and we skip n if $n = 1$. The differential is given by the formula

$$d(n\gamma_1|\cdots|\gamma_s) = \psi_N(n)\gamma_1|\cdots|\gamma_s + \sum_{i=1}^{s}(-1)^i n\gamma_1|\cdots|\Delta(\gamma_i)|\cdots|\gamma_s + (-1)^{s+1}n\gamma_1|\cdots|\gamma_s|1$$

where ψ_N is the right Γ-comodule map for N. It is convenient to note that the differentials are determined by

$$d(n) = \psi_N(n) - n, \quad d(\gamma) = 1|\gamma + \gamma|1 - \Delta(\gamma) \tag{2}$$

and extended to other elements of the cobar complex by requiring that d be a graded derivation with respect to the tensor product grading as follows:

$$d(n\gamma_1|\cdots|\gamma_s) = d(n)\gamma_1|\cdots|\gamma_s + \sum_{i=1}^{s}(-1)^{i-1}n\gamma_1|\cdots|d(\gamma_i)|\cdots|\gamma_s.$$

When $N = A$ then the complex (1) for computing $\mathrm{Ext}(A)$ can be identified with

$$A \xrightarrow{d} \Gamma \xrightarrow{d} \Gamma \otimes_A \Gamma \longrightarrow \cdots \tag{3}$$

with $d(a) = \eta_R(a) - a$ and $d(\gamma) = 1|\gamma + \gamma|1 - \Delta(\gamma)$.

Eventually, if N is a comodule algebra (like A), then $\mathrm{Ext}(N)$ is equipped with a product [5, A.1.2.15]. In case when $N = A$ this product is represented in the cobar construction by concatenation

$$\gamma_1|\cdots|\gamma_s \times \gamma'_1|\cdots|\gamma'_r = \gamma_1|\cdots|\gamma_s|\gamma'_1|\cdots|\gamma'_r. \tag{4}$$

Next we shall review the structure of the Hopf algebroid (BP_*, BP_*BP) associated with the Brown-Peterson spectrum BP for a fixed prime p. These formulae and Quillen's construction of the spectrum BP can be found in [5, Ch.4] and the relation with formal group laws is described in [5, Appendix A2]. A good introduction to the subject is [6].

We have the isomorphisms of graded rings

$$BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots], \quad |v_i| = 2(p^i - 1),$$
$$BP_*BP = BP_*[t_1, t_2, \ldots], \quad |t_i| = 2(p^i - 1).$$

The structural maps of the Hopf algebroid are given in terms of the rationalization $BP_* \otimes \mathbb{Q} = \mathbb{Q}[l_1, l_2, \ldots]$ where $|l_i| = 2(p^i - 1)$. In terms of the formal group laws the l_i are the coefficients of the logarithm for the universal p-typical formal group law. The generators of BP_* and $BP_* \otimes \mathbb{Q}$ are related by Hazewinkel relations:

$$pl_n = \sum_{0 \le i < n} l_i v_{n-i}^{p^i}, \quad l_0 = v_0 = 1. \tag{5}$$

The left unit is just the inclusion $\eta_L : BP_* \to BP_*BP$ and the right unit is given on the generators of $BP_* \otimes \mathbb{Q}$ by

$$\eta_R(l_n) = \sum_{0 \le i \le n} l_i t_{n-i}^{p^i} \tag{6}$$

and the diagonal of BP_*BP is determined by:

$$\sum_{i,j} l_i \Delta(t_j)^{p^i} = \sum_{i,j,k} l_i t_j^{p_i} \otimes t_k^{p^{j+k}}. \tag{7}$$

These equations determine the Hopf algebroid structure uniquely.

Consider the invariant ideals $I_n = (p, v_1 \ldots, v_{n-1}) \subset BP_*$ and $I = (p, v_1, v_2, \ldots) \subset BP_*$. The following relations, which hold in (BP_*, BP_*BP), can be easily derived from (5)-(7), see eg. [4, B.5.15] or [5] for a proof.

$$\begin{aligned}
&(8) & \eta_R(v_1) &= v_1 + pt_1 \\
&(9) & \eta_R(v_n) &\equiv v_n \pmod{I_n} \\
&(10) & \eta_R(v_{n+j}) &\equiv v_{n+j} + v_n t_j^{p^n} - v_n^{p^j} t_j \pmod{I_n, t_1, \ldots, t_{j-1}}, \quad j \ge 1 \\
&(11) & \Delta(t_1) &= 1|t_1 + t_1|1 \\
&(12) & \Delta(t_2) &= 1|t_2 + t_2|1 - v_1 t_1|t_1 + t_1|t_1^2 \text{ for } p = 2 \\
&(13) & \eta_R(v_2) &= v_2 - 5v_1 t_1^2 - 3v_1^2 t_1 + 2t_2 - 4t_1^3 \text{ for } p = 2
\end{aligned}$$

We conclude by making a notational convention: in the sequel, unless indicated otherwise, **the pair** (A, Γ) **always denotes** (BP_*, BP_*BP).

2. Calculating $\mathrm{Ext}^1(A)$

This calculation follows precisely the strategy outlined in [6]. First, since $I_n \subset A$ are invariant ideals, we can consider the Γ-comodules A/I_n for $n \ge 0$. We start by computing Ext^0 for these comodules.

Lemma 2.1 (Landweber, [1])**.**

$$Ext^0(A) = \mathbb{Z}_{(p)}, \quad Ext^0(A/I_n) = \mathbb{F}_p[v_n]$$

Proof. By definition, $\mathrm{Ext}^0(A/I_n)$ is the kernel of $(\eta_R - \mathrm{id}) : A/I_n \to \Gamma/I_n$, so it contains all powers of v_n by (9). Now suppose x is an element in A/I_n such that $\eta_R(x) - x \equiv 0 \pmod{I_n}$. Let v_{n+j} be the highest of the v which appear in x and write x as

$$x = v_{n+j}^l f_l + \ldots + v_{n+j} f_1 + f_0,$$

where each f_i is a polynomial in $v_n, \ldots, v_{n+j-1}$. Let $J_{n,j}$ be the ideal $(p, v_1, \ldots, v_{n-1}, t_1, \ldots, t_{j-1}) \subset \Gamma$. By (10) we have the following relations modulo $J_{n,j}$:

$$\begin{aligned}
\eta_R(v_{n+j}) &\equiv v_{n+j} + v_n t_j^{p^n} - v_n^{p^j} t_j \pmod{J_{n,j}}, \\
\eta_R(v_k) &\equiv v_k \pmod{J_{n,j}} \text{ for } k = n, \ldots, n+j-1.
\end{aligned}$$

The second one implies that $\eta_R(f_i) \equiv f_i \pmod{J_{n,j}}$, so modulo this ideal we have a congruence

$$0 \equiv \eta_R(x) - x \equiv \sum_{i=0}^{l} ((v_{n+j} + v_n t_j^{p^n} - v_n^{p^j} t_j)^i f_i - v_{n+j}^i f_i) \pmod{J_{n,j}},$$

but the sum equals $v_n^l t_j^{lp^n} f_l +$ (lower powers of t_j), so it follows that $f_l \equiv 0 \pmod{I_n}$. A repeated application of this argument proves that x is a monomial in v_n. □

We can now describe the generators of $\mathrm{Ext}^1(A)$. First of all, since $\mathrm{Ext}^i(A \otimes \mathbb{Q}) = 0$ for $i > 0$ (for a proof see [5, Thm. 5.2.1]), and since (A, Γ) is $\mathbb{Z}_{(p)}$-local, all $\mathrm{Ext}^i(A)$ are p-torsion groups for all $i > 0$. The short exact sequence of Γ-comodules

$$0 \longrightarrow A \xrightarrow{\cdot p} A \longrightarrow A/(p) \longrightarrow 0$$

yields a long sequence of Ext groups
(14)

$$0 \longrightarrow \mathrm{Ext}^0(A) \xrightarrow{\cdot p} \mathrm{Ext}^0(A) \longrightarrow \mathrm{Ext}^0(A/(p)) \xrightarrow{\delta} \mathrm{Ext}^1(A) \xrightarrow{\cdot p} \mathrm{Ext}^1(A) \longrightarrow \cdots.$$

By Lemma 2.1 for $n = 1$ it becomes

$$(15) \qquad 0 \longrightarrow \mathbb{Z}_{(p)} \xrightarrow{\cdot p} \mathbb{Z}_{(p)} \longrightarrow \mathbb{F}_p[v_1] \xrightarrow{\delta} \mathrm{Ext}^1(A) \xrightarrow{\cdot p} \mathrm{Ext}^1(A) \longrightarrow \cdots.$$

We define elements $\alpha_t \in \mathrm{Ext}^{1,2t(p-1)}(A)$ by $\alpha_t = \delta(v_1^t)$. By exactness of (15) the elements α_t are nonzero, have order p and each group $\mathrm{Ext}^{1,2t(p-1)}(A)$ is a cyclic p-group, while all other groups in $\mathrm{Ext}^{1,*}(A)$ are trivial. It remains to determine the orders of the groups, i.e. the divisibility of α_t by p.

We can perform this calculation in the cobar construction and write explicit representatives for α_t. We have the diagram in which each column is the cobar resolution for the respective comodule:

$$(16) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\cdot p} & A & \longrightarrow & A/(p) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \eta_R - \mathrm{id}} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Gamma & \xrightarrow{\cdot p} & \Gamma & \longrightarrow & \Gamma/(p) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \cdots & & \cdots & & \cdots & & \end{array}$$

Applying the definition of the connecting homomorphism δ and (8) we obtain

$$(17) \qquad \alpha_t = \delta(v_1^t) = \frac{1}{p}(\eta_R(v_1^t) - v_1^t) = \frac{1}{p}((v_1 + pt_1)^t - v_1^t)$$

From now on we fix a factorization $t = sp^i$ where s is not divisible by p.

If p is odd then we have

$$\alpha_t = sp^i v_1^{sp^i - 1} t_1 + (\text{terms divisible by } p^{i+1})$$

and when $p = 2$ we have

$$\alpha_t = s2^i v_1^{s2^i - 1} t_1 + s2^i(s2^i - 1) v_1^{s2^i - 2} t_1^2 + (\text{terms divisible by } 2^{i+1}).$$

In either case α_t is certainly divisible by p^i and we define $\alpha_{t/j} = \alpha_t/p^{j-1}$ for $j = 1, \dots, i+1$, so that $\alpha_{t/j}$ is an element of order p^j in $\mathrm{Ext}^1(A)$.

If p is odd then

$$\alpha_{t/i+1} = sv_1^{sp^i-1}t_1 + (\text{terms divisible by } p).$$

It follows from the portion of the long exact sequence

$$\mathrm{Ext}^1(A) \xrightarrow{\cdot p} \mathrm{Ext}^1(A) \longrightarrow \mathrm{Ext}^1(A/(p))$$

that an element of $\mathrm{Ext}^1(A)$ is divisible by p if and only if its image in $\mathrm{Ext}^1(A/(p))$ is zero. The image of $\alpha_{t/i+1}$ in $\mathrm{Ext}^1(A/(p))$ is represented by $sv_1^{sp^i-1}t_1$ and it is nonzero by Lemma 2.3.i. Therefore $\alpha_{t/i+1}$ is the generator of $\mathrm{Ext}^{1,2(p-1)sp^i}(A) \simeq \mathbb{Z}/(p^{i+1})$.

If $p = 2$ the situation is more complicated.

- If t is odd (i.e. $i = 0$) we have

  $$\alpha_{t/1} = v_1^{s-1}t_1 + (\text{terms divisible by } 2).$$

  As before, the mod 2 reduction of this element is $v_1^{s-1}t_1$ which is nonzero in $\mathrm{Ext}^1(A/(2))$ by Lemma 2.3.i, hence $\mathrm{Ext}^{1,2s}(A) \simeq \mathbb{Z}/(2)$.
- If $t = 2$ (i.e. $s = 1$, $i = 1$ and we are in $\mathrm{Ext}^{1,4}(A)$) then we have exactly

  $$\alpha_{2/2} = v_1t_1 + t_1^2.$$

  Note that in dimension 4 the image $d(A/(2)) \subset \Gamma/(2)$ is spanned by $\eta_R(v_1^2) - v_1^2 = 4(v_1t_1 + t_1^2) \equiv 0$, so $\alpha_{2/2}$ necessarily gives a nonzero element in $\mathrm{Ext}^1(A/(2))$. It means that $\mathrm{Ext}^{1,4}(A) \simeq \mathbb{Z}/(4)$.
- If $t \geq 4$ is even (i.e. $i \geq 1$) then

  $$\alpha_{t/i+1} = v_1^{t-1}t_1 + v_1^{t-2}t_1^2 + (\text{terms divisible by } 2)$$

  The image of $\alpha_{t/i+1}$ in $\mathrm{Ext}^1(A/(2))$ is $v_1^{t-1}t_1 + v_1^{t-2}t_1^2$, but this time it is trivial. Indeed, using (9), (10) we have

  $$\eta_R(v_1) \equiv v_1 \pmod 2, \quad \eta_R(v_2) \equiv v_2 + v_1t_1^2 + v_1^2t_1 \pmod 2,$$

  and we easily see that

  $$\alpha_{t/i+1} \equiv \eta_R(v_2v_1^{t-3}) - v_2v_1^{t-3} = d(v_2v_1^{t-3}) \pmod 2.$$

  It means that the element $\alpha_{t/i+1} + (\eta_R(v_2v_1^{t-3}) - v_2v_1^{t-3})$ in divisible by 2 in Γ, so we can define

  $$\begin{aligned} \alpha_{t/i+2} &= \frac{1}{2}\big(\alpha_{t/i+1} + (\eta_R(v_2v_1^{t-3}) - v_2v_1^{t-3})\big) \\ &= \frac{\frac{1}{2^{i+1}}((v_1+2t_1)^t - v_1^t) + (\eta_R(v_2v_1^{t-3}) - v_2v_1^{t-3})}{2}. \end{aligned} \tag{18}$$

  In order to prove that this element is not further divisible by 2 we compute its image in $\mathrm{Ext}^1(A/(2))$, which is equivalent to computing the numerator of (18) mod 4. We don't need an exact formula; it suffices to recall from (13) that

  $$\eta_R(v_2) \equiv v_2 - 5v_1t_1^2 - 3v_1^2t_1 + 2t_2 \pmod 4$$

  and it quickly follows that the image of $\alpha_{t/i+2}$ in $\mathrm{Ext}^1(A/(2))$ has the form

  $$\alpha_{t/i+2} \equiv t_2v_1^{t-3} + (\text{terms divisible by } t_1) \pmod 2$$

and a cocycle of this form is nonzero in $\mathrm{Ext}^1(A/(2))$ by Lemma 2.3.ii.

All of this can be summarized in the following theorem [5, 5.2.6].

Theorem 2.2. *Let $t = sp^i$. The generator of $Ext^{1,2(p-1)t}(A)$ is*

$$\alpha_{t/i+1} = \frac{1}{p^{i+1}}(\eta_R(v_1^t) - v_1^t) = \frac{1}{p^{i+1}}((v_1 + pt_1)^t - v_1^t)$$

unless $p = 2$ and $t \neq 4$ is even, when the generator of $Ext^{1,2t}(A)$ is $\alpha_{t/i+2}$ given by (18). It follows that

$$Ext^{1,2(p-1)t}(A) = \mathbb{Z}/(p^{i+1})$$

unless $p = 2$ and $t \neq 4$ is even, when

$$Ext^{1,2t}(A) = \mathbb{Z}/(2^{i+2}).$$

It remains to prove the following lemma, which identifies some nonzero elements in $\mathrm{Ext}^1(A/(p))$. The proof is similar to that of Lemma 2.1.

Lemma 2.3. *For any prime p:*

i) *The cocycle $v_1^k t_1 \in \Gamma/(p)$ represents a nonzero element in $Ext^1(A/(p))$.*

ii) *If $\alpha \in \Gamma/(p)$ is a cocycle such that*

$$\alpha \equiv t_2 v_1^k \pmod{(p, t_1)}$$

then α represents a nonzero element in $Ext^1(A/(p))$.

Proof. We first prove (i). Suppose that $x \in A/(p)$ is an element such that

$$\eta_R(x) - x = v_1^k t_1 \text{ in } \Gamma/(p).$$

Then, in particular

$$\eta_R(x) - x \equiv 0 \pmod{(p, t_1)}.$$

Now we use an argument identical to that of Lemma 2.1. Let v_{1+j} be the highest v occurring in x and note, after (10):

$$\begin{aligned} \eta_R(v_{1+j}) &= v_{1+j} + v_1 t_j^p - v_1^{p^j} t_j \pmod{(p, t_1, \ldots, t_{j-1})} \\ \eta_R(v_k) &= v_k \pmod{(p, t_1, \ldots, t_{j-1})} \text{ for } k \leq j. \end{aligned}$$

If $j \geq 2$ we argue as in Lemma 2.1 that the coefficient at the highest power of v_{1+j} in x must be 0. Therefore x is a polynomial in v_1, v_2 only. Write x as

$$x = v_2^l f_l + \ldots + v_2 f_1 + f_0$$

where each f_i is a multiple of an appropriate power of v_1. Computing modulo p we now have

$$v_1^k t_1 \equiv \eta_R(x) - x \equiv \sum_{i=0}^{l} ((v_2 + v_1 t_1^p - v_1^p t_1)^i f_i - v_2^i f_i)$$

which is $v_1^l t_1^{lp} f_l +$ (lower powers of t_1), so we must have $f_l \equiv 0 \pmod{p}$ for $l \geq 1$. An inductive repetition proves that $x = f_0$ but then $\eta_R(x) - x \equiv 0 \pmod{p}$. This means we have a contradiction.

Now we move to (ii). The argument is similar, but one step longer. Suppose $\eta_R(x) - x = \alpha$ in $\Gamma/(p)$. Then $\eta_R(x) - x \equiv 0 \pmod{(p, t_1, t_2)}$ and the same method proves that x is a polynomial in v_1, v_2, v_3. The presence of v_3 is eliminated by computing modulo (p, t_1) and eliminating excessive powers of t_2 which originate from $\eta_R(v_3) \equiv v_3 + v_1 t_2^p - v_1^{p^2} t_2 \pmod{(p, t_1)}$. It remains to consider the case when x is a polynomial in v_1, v_2. Note that

$$\eta_R(v_1) \equiv v_1 \pmod{(p,t_1)}, \quad \eta_R(v_2) \equiv v_2 \pmod{(p,t_1)},$$

hence $\eta_R(x) - x \equiv 0 \pmod{(p,t_1)}$ for any such element x, which contradicts $\eta_R(x) - x \equiv t_2 v_1^k \pmod{(p,t_1)}$. □

3. Hopf invariant one

In this, and the following sections, we perform some simple explicit calculations with the elements we have just defined. We begin with the relation between the Adams-Novikov spectral sequence for BP and the classical Adams mod p spectral sequence for $H/(p)$.

The Brown-Peterson spectrum BP comes with a map $\Theta : BP \to H/(p)$ to the mod p Eilenberg-MacLane spectrum $H/(p)$, which induces a map from the Adams-Novikov spectral sequence to the classical Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_p^*}(\mathbb{F}_p, \mathbb{F}_p) \implies \pi_{s-t}^S(S^0)_{(p)}.$$

where $\mathcal{A}_p^*$ denotes the mod p dual Steenrod algebra. For our purposes it will suffice to describe the map of Hopf algebroids

$$(A, \Gamma) \to (\mathbb{F}_p, \mathcal{A}_p^*)$$

induced by Θ. Let us remind the structure of $\mathcal{A}_p^*$ as a Hopf algebra. When $p = 2$ we have

$$\mathcal{A}_2^* = \mathbb{F}_2[\xi_1, \xi_2, \ldots], \quad |\xi_i| = 2^i - 1$$

and for odd p:

$$\mathcal{A}_p^* = \mathbb{F}_p[\xi_1, \xi_2, \ldots] \otimes \Lambda[\tau_0, \tau_1, \ldots], \quad |\xi_i| = 2(p^i - 1), |\tau_i| = 2p^i - 1.$$

In each case the diagonal is given on the polynomial part by

$$\Delta(\xi_k) = \sum_{0 \le i \le k} \xi_{k-i}^{p^i} | \xi_i, \tag{19}$$

which, in the case $p = 2$ implies also

$$\Delta(\xi_k^2) = \sum_{0 \le i \le k} (\xi_{k-i}^2)^{p^i} | \xi_i^2, \tag{20}$$

It follows from the defining relations (5)-(7) that the diagonal of (A, Γ) satisfies

$$\Delta(t_k) \equiv \sum_{0 \le i \le k} t_i | t_{k-i}^{p^i} \pmod{I} \tag{21}$$

(see eg. [4, B.5.15]). For odd p (19) and (21) imply that the assignment $v_i \mapsto 0, t_i \mapsto \xi_i$ extends to a map of Hopf algebroids from (A, Γ) to the opposite Hopf algebra $(\mathbb{F}_p, \overline{\mathcal{A}_p^*})$, or, in other words, the assignment

$$v_i \mapsto 0, t_i \mapsto c(\xi_i)$$

where c is the conjugation of $\mathcal{A}_p^*$, extends to a map of Hopf algebroids $(A, \Gamma) \to (\mathbb{F}_p, \mathcal{A}_p^*)$. In a similar fashion, (20) and (21) imply that for $p = 2$ the assignment

$$v_i \mapsto 0, t_i \mapsto c(\xi_i^2)$$

extends to a map of Hopf algebroids $(A, \Gamma) \to (\mathbb{F}_2, \mathcal{A}_2^*)$. In each case this is the map induced by the map of spectra $BP \to H/(p)$.

It follows that computing the image of the map

$$\mathrm{Ext}_\Gamma(A,A)\to \mathrm{Ext}_{\mathcal{A}_p^*}(\mathbb{F}_p,\mathbb{F}_p) \tag{22}$$

is equivalent to computing the reduction mod I in $\mathrm{Ext}_\Gamma(A,A)$ and substituting $t_i\mapsto c(\xi_i)$ for odd p or $t_i\mapsto c(\xi_i^2)$ for $p=2$. Note, in particular, that the Adams 1-line $\mathrm{Ext}^{1,*}_{\mathcal{A}_p^*}(\mathbb{F}_p,\mathbb{F}_p)$ is generated by the elements $h_i=[\xi_1^{p^i}]$ (and an additional $a_0=[\tau_0]$ for odd p). Moreover, we have $c(\xi_1)=-\xi_1$ and $c(\xi_1^2)=\xi_1^2$ in $\mathcal{A}_p^*$.

We will show that for $p=2$ the only elements in $\mathrm{Ext}^{1,*}$ with nonzero image in the Adams spectral sequence are $\alpha_1\in\mathrm{Ext}^{1,2}$, $\alpha_{2/2}\in\mathrm{Ext}^{1,4}$ and $\alpha_{4/4}\in\mathrm{Ext}^{1,8}$ with images h_1,h_2,h_3. It is the content of [5, Thm.5.2.8]. This follows from a direct reduction mod I (which we denote $\equiv_I$) using the formulas of Theorem 2.2.

$$\begin{aligned}
\alpha_1 =& t_1\mapsto \xi_1^2=h_1,\\
\alpha_{2/2} =& \frac{1}{4}((v_1+2t_1)^2-v_1^2)\equiv_I t_1^2\mapsto \xi_1^4=h_2\\
\alpha_{4/4} =& \frac{1}{2}(\frac{1}{8}((v_1+2t_1)^4-v_1^4)-(\eta_R(v_2v_1)-v_2v_1))\equiv_I t_1^4\mapsto \xi_1^8=h_3.
\end{aligned}$$

All the remaining generators of $\mathrm{Ext}^1(A)$ are mapped to zero. Indeed, we only need to check this for $\alpha_{2^i/i+2}$ where $i\ge 3$. In that case $\alpha_{2^i/i+2}$ is given by (18) with $t=2^i$. By (13) we have $\eta_R(v_2)\equiv 2t_2 \pmod{4,v_1,v_2,\dots}$, hence

$$\alpha_{2^i/i+2}\equiv_I 2^{2^i-i-2}t_1+2^{2^i-3}t_2t_1\equiv_I 0.$$

If p is odd and $t=sp^i>1$ the image of the generator $\alpha_{t/i+1}\in\mathrm{Ext}^{1,2(p-1)sp^i}$ in the mod p Adams spectral sequence is zero, because

$$\alpha_{t/i+1}=\frac{1}{p^{i+1}}((v_1+pt_1)^t-v_1^t)\equiv_I p^{sp^i-i-1}t_1^{sp^i}\equiv_I 0.$$

When $t=1$ the image of α_1 is

$$\alpha_1=\frac{1}{p}((v_1+pt_1)-v_1)\equiv_I t_1\mapsto\xi_1=h_0.$$

4. The β-family in Ext^2

The elements β_t in Ext^2 are defined as images of v_2^t under the composition

$$\mathbb{F}_p[v_2]=\mathrm{Ext}^0(A/(p,v_1))\xrightarrow{\delta^1}\mathrm{Ext}^1(A/(p))\xrightarrow{\delta^0}\mathrm{Ext}^2(A) \tag{23}$$

where δ^n is the connecting homomorphism corresponding to the short exact sequence of comodules

$$0\longrightarrow A/I_n\xrightarrow{\cdot v_n}A/I_n\longrightarrow A/I_{n+1}\longrightarrow 0\ . \tag{24}$$

In particular, at the level of cobar constructions, we have

$$\delta^1(v_2^t)=\frac{1}{v_1}(\eta_R(v_2^t)-v_2^t)\in\Gamma/(p).$$

We also denote this element by $\beta_t \in \mathrm{Ext}^1(A/(p))$ and we let $\beta_{t/i} \in \mathrm{Ext}^1(A/(p))$ to be defined by the condition $\beta_{t/i} \cdot v_1^{i-1} = \beta_t$, whenever it exists. Recalling from (10) that $\eta_R(v_2) \equiv v_2 + v_1 t_1^p - v_1^p t_1 \pmod p$, we obtain the formula for a representative in $\Gamma/(p)$:

$$\beta_t = \delta^1(v_2^t) = \frac{1}{v_1}((v_2 + v_1 t_1^p - v_1^p t_1)^t - v_2^t)$$

which is particularly useful when $t = p^i$. We then pass to $\mathrm{Ext}^2(A)$ defining elements $\beta_{t/i} \in \mathrm{Ext}^2(A)$ as images of $\beta_{t/i} \in \mathrm{Ext}^1(A/(2))$ under δ^0 and then $\beta_{t/i,j} \in \mathrm{Ext}^2(A)$ by the condition $\beta_{t/i,j} \cdot p^{j-1} = \beta_{t/i}$. In short:

$$\beta_{t/i,j} = \frac{1}{p^{j-1}}\delta^0(\frac{1}{v_1^{i-1}}\delta^1(v_2^t)) \tag{25}$$

whenever the right-hand side makes sense. As usually we also abbreviate $\beta_{t/i} = \beta_{t/i,1}$.

We will now find the representatives for some of the elements of the β-family (namely $\beta_{2^j/2^j}$, $\beta_{2^j/2^j-1}$ and $\beta_{4/2,2}$), together with their images in the Adams spectral sequence. This will prove part of [5, Thm.5.4.6]. In particular, with $\beta_{4/2,2}$ we will experience similar divisibility issues we had in the 1-line. Before we start, let us make the following observation which will simplify some of the calculations.

Lemma 4.1. *If $j \geq 1$ and $a \geq p^{j-1}$, the cocycles $v_1^a t_1^{p^j}$ and $v_1^{a+p^j-1} t_1$ represent the same element of $Ext^1(A/(p))$.*

Proof. Suppose $n \geq 0$ and m is a power of p. Using

$$\eta_R(v_1) \equiv v_1 \pmod p, \quad \eta_R(v_2) \equiv v_2 + v_1 t_1^p - v_1^p t_1 \pmod p,$$

we obtain

$$\begin{aligned} \eta_R(v_1^n v_2^m) - v_1^n v_2^m &\equiv v_1^n (v_2 + v_1 t_1^p - v_1^p t_1)^m - v_1^n v_2^m \\ &\equiv v_1^{n+m} t_1^{pm} - v_1^{n+pm} t_1^m \pmod p \end{aligned}$$

which means that $v_1^{n+m} t_1^{pm}$ and $v_1^{n+pm} t_1^m$ represent the same element of $\mathrm{Ext}^1(A/(p))$. The result follows by repeated application of this fact. □

We proceed with the description of a few selected elements of the β-family.

- **$\beta_{2^j/2^j}$ for $p = 2$.** The image of $\delta^1(v_2^{2^j})$ is represented in $\Gamma/(2)$ by

$$\beta_{2^j} \equiv \frac{1}{v_1}((v_2 + v_1 t_1^2 + v_1^2 t_1)^{2^j} - v_2^{2^j}) \equiv v_1^{2^j-1} t_1^{2^{j+1}} + v_1^{2^{j+1}-1} t_1^{2^j}.$$

From this we get an element $\beta_{2^j/2^j} \in \mathrm{Ext}^1(A/(2))$ represented in $\Gamma/(2)$ by

$$\beta_{2^j/2^j} = \frac{1}{v_1^{2^j-1}} \beta_{2^j} = t_1^{2^{j+1}} + v_1^{2^j} t_1^{2^j}.$$

By Lemma 4.1, the same cohomology class in $\mathrm{Ext}^1(A/(2))$ is represented by

$$\beta_{2^j/2^j} = t_1^{2^{j+1}} + v_1^{2^{j+1}-1} t_1.$$

It follows that the element $\beta_{2^j/2^j} \in \mathrm{Ext}^2(A)$ is represented by

$$\begin{aligned}\beta_{2^j/2^j} &= \delta^0(t_1^{2^{j+1}} + v_1^{2^{j+1}-1}t_1)\\ &= \frac{1}{2}\big(d(t_1^{2^{j+1}}) + d(v_1^{2^{j+1}-1})|t_1 + v_1^{2^{j+1}-1}d(t_1)\big).\end{aligned}$$

From (11) we have $d(t_1)=0$ and $d(t_1^k) = 1|t_1^k + t_1^k|1 - (1|t_1+t_1|1)^k$, so

$$\beta_{2^j/2^j} = \frac{1}{2}\Big(\sum_{0<i<2^{j+1}} \binom{2^{j+1}}{i} t_1^i|t_1^{2^{j+1}-i} + ((v_1+2t_1)^{2^{j+1}-1} - v_1^{2^{j+1}-1})|t_1\Big).$$

The only binomial coefficient in the 2^{j+1}-th row of the Pascal triangle which is divisible only by 2 but not by 4 is the middle one. It follows that

$$\beta_{2^j/2^j} = t_1^{2^j}|t_1^{2^j} + v_1^{2^{j+1}-2}t_1|t_1 + (\text{terms divisible by } 2).$$

It means that $\beta_{2^j/2^j} \equiv_I t_1^{2^j}|t_1^{2^j}$, and this element maps to $\xi_1^{2^{j+1}}|\xi_1^{2^{j+1}} = h_{j+1}^2$ in the Adams spectral sequence.

- β_{p^j/p^j} **for odd** p**.** For comparison, we see that an analogous calculation shows that $\beta_{p^j/p^j} \in \mathrm{Ext}^1(A/(p))$ is represented by

$$\beta_{p^j/p^j} = t_1^{p^{j+1}} + v_1^{p^{j+1}-1}t_1.$$

It follows from our calculation of $\mathrm{Ext}^1(A)$ that $v_1^{p^{j+1}-1}t_1$ is the mod p reduction of $\alpha_{p^{j+1}/j+2} \in \mathrm{Ext}^1(A)$. The portion of the long exact sequence (14)

$$\mathrm{Ext}^1(A) \longrightarrow \mathrm{Ext}^1(A/(p)) \xrightarrow{\delta^0} \mathrm{Ext}^2(A)$$

now implies that $\delta^0(v_1^{p^{j+1}-1}t_1) = 0$, so in $\mathrm{Ext}^2(A)$ we have

$$\beta_{p^j/p^j} = \delta^0(t_1^{p^{j+1}}) = \frac{1}{p}\Big(\sum_{0<i<p^{j+1}} \binom{p^{j+1}}{i} t_1^i|t_1^{p^{j+1}-i}\Big).$$

- $\beta_{2^j/2^j-1}$ **for** $p=2$**.** Following the previous calculation we get in $\mathrm{Ext}^1(A/(2))$

$$\beta_{2^j/2^j-1} = \frac{1}{v_1^{2^j-2}}\beta'_{2^j} = v_1t_1^{2^{j+1}} + v_1^{2^j-1}t_1^{2^j}.$$

By Lemma 4.1 this element is also represented by $v_1t_1^{2^{j+1}} + v_1^{2^{j+1}}t_1$, hence in $\mathrm{Ext}^2(A)$ we have

$$\begin{aligned}\beta_{2^j/2^j-1} &= \frac{1}{2}d(v_1t_1^{2^{j+1}} + v_1^{2^{j+1}}t_1)\\ &= \frac{1}{2}\big(d(v_1)|t_1^{2^{j+1}} + v_1|d(t_1^{2^{j+1}}) + d(v_1^{2^{j+1}})|t_1\big)\\ &= \frac{1}{2}\Big(2t_1|t_1^{2^{j+1}} + v_1|d(t^{2^{j+1}}) + ((v_1+2t_1)^{2^{j+1}} - v_1^{2^{j+1}})|t_1\Big)\\ &\equiv_I t_1|t_1^{2^{j+1}}\end{aligned}$$

which maps to $\xi_1^2|\xi_1^{2^{j+2}} = h_1h_{j+2}$ in the Adams spectral sequence.

- **$\beta_{4/2,2}$ for $p = 2$.** First, we find $\beta_4 \in \mathrm{Ext}^1(A/(2))$:

$$\beta_4 = \delta^1(v_2^4) = \frac{1}{v_1}\big((v_2 + v_1 t_1^2 - v_1^2 t_1)^4 - v_2^4\big) \equiv v_1^3 t_1^8 + v_1^7 t_1^4 \pmod 2$$

so we can consider the element

$$\beta_{4/2} = \frac{1}{v_1}\beta_4 = v_1^2 t_1^8 + v_1^6 t_1^4.$$

Its image in $\mathrm{Ext}^2(A)$ is

$$\begin{aligned}\beta_{4/2} &= \frac{1}{2}(d(v_1^2)|t_1^8 + v_1^2 d(t_1^8) + d(v_1^6)|t_1^4 + v_1^6 d(t_1^4)) \\ &= \frac{1}{2}\Big((4v_1t_1 + 4t_1^2)|t_1^8 + v_1^2 \sum_{0<i<8}\binom{8}{i} t_1^i|t_1^{8-i} + \\ &\qquad + ((v_1+2t_1)^6 - v_1^6)|t_1^4 + v_1^6 \sum_{0<i<4}\binom{4}{i} t_1^i|t_1^{4-i}\Big) \\ &= v_1^2 t_1^4|t_1^4 + v_1^6 t_1^2|t_1^2 + (\text{terms divisible by } 2)\end{aligned}$$

In order to see that $\beta_{4/2,2}$ exists, we must know that $\beta_{4/2}$ is divisible by 2 or, equivalently, its reduction to $\mathrm{Ext}^2(A/(2))$ is zero. This time it is not so easy to guess the expression as a coboundary, but it turns out that

(26) $$v_1^2 t_1^4|t_1^4 + v_1^6 t_1^2|t_1^2 \equiv d(v_2^2 t_1^4 + v_1^4 t_2^2) \pmod 2.$$

The proof is straightforward using the mod 2 reductions of (8)-(13):

$$d(v_2^2) \equiv v_1^2 t_1^4 + v_1^4 t_1^2, \quad d(t_1^4) \equiv 0,$$
$$d(v_1^4) \equiv 0, \quad d(t_2^2) \equiv v_1^2 t_1^2|t_1^2 + t_1^2|t_1^4.$$

Therefore we can define

$$\beta_{4/2,2} = \frac{1}{2}(\beta_{4/2} + d(v_2^2 t_1^4 + v_1^4 t_2^2))$$

and after a rather tedious, but straightforward calculation we get

$$\beta_{4/2,2} \equiv t_1^2|t_1^8 + v_2^2 t_1^2|t_1^2 \pmod{2, v_1}$$

so $\beta_{4,2/2} \equiv_I t_1^2|t_1^8$ and its image in the Adams spectral sequence is $\xi_1^4|\xi_1^{16} = h_2h_4$.

5. An example in higher Ext groups

As a final example we prove [5, Prop.5.1.21]. First, recall that as a generalization of the α- and β-families, we define the n-th Greek letter element $\alpha_t^{(n)}$ as the image of v_n^t under the composition of connecting homomorphisms of (24)

(27) $$\mathbb{F}_p[v_n] = \mathrm{Ext}^0(A/I_n) \xrightarrow{\delta^{n-1}} \mathrm{Ext}^1(A/I_{n-1}) \xrightarrow{\delta^{n-2}} \cdots \xrightarrow{\delta^0} \mathrm{Ext}^n(A)$$

so that, for example, $\alpha_t = \alpha_t^{(1)}$, $\beta_t = \alpha_t^{(2)}$, etc. The proposition we want to prove is

(28) $$\alpha_1^{(n+1)} = -\alpha_{p-1}^{(n)}\alpha_1 \quad \text{for } n \geq 2.$$

(It is equivalent with the formulation of [5, Prop.5.1.21] using graded commutativity of product in Ext.)

First, observe that every connecting homomorphism $\delta^k : \mathrm{Ext}^*(A/I_{k+1}) \to \mathrm{Ext}^{*+1}(A/I_k)$ of (27) satisfies

$$\delta^k(x|t_1) = \delta^k(x)|t_1$$

which follows from

$$\delta^k(x|t_1) = \frac{1}{v_k}d(x|t_1) = \frac{1}{v_k}(d(x)|t_1 \pm x|d(t_1)) = (\frac{1}{v_k}d(x))|t_1 = \delta^k(x)|t_1$$

because $d(t_1) = 0$.

Now we can calculate the element $\alpha_1^{(n+1)}$. We start with $v_{n+1} \in \mathbb{F}_p[v_{n+1}] = \mathrm{Ext}^0(A/I_{n+1})$ and its image in $\mathrm{Ext}^1(A/I_n)$:

$$\begin{aligned}\delta^n(v_{n+1}) &= \frac{1}{v_n}(\eta_R(v_{n+1}) - v_{n+1}) \\ &\equiv \frac{1}{v_n}(v_n t_1^{p^n} - v_n^p t_1) = t_1^{p^n} - v_n^{p-1}t_1 \pmod{I_n}\end{aligned}$$

The image of this element in $\mathrm{Ext}^2(A/I_{n-1})$ is in turn given by

$$\begin{aligned}\delta^{n-1}\delta^n(v_{n+1}) &= \frac{1}{v_{n-1}}\big(d(t_1^{p^n}) - d(v_n^{p-1})|t_1 - v_n^{p-1}|d(t_1)\big) \\ &= -\frac{d(v_n^{p-1})}{v_{n-1}}|t_1 = -\delta^{n-1}(v_n^{p-1})|t_1\end{aligned}$$

because $d(t_1^{p^n}) = \sum_{0<i<p^n}\binom{p^n}{i}t_1^i|t_1^{p^n-i} \equiv 0 \pmod{p}$ and $(p) \subset I_{n-1}$. We now apply all the remaining connecting homomorphisms δ to obtain

$$\begin{aligned}\delta^0\delta^1\cdots\delta^{n-1}\delta^n(v_{n+1}) &= -\delta^0\cdots\delta^{n-2}\big(\delta^{n-1}(v_n^{p-1})\big)|t_1) \\ &= -\delta^0\cdots\delta^{n-2}\delta^{n-1}(v_n^{p-1})|t_1\end{aligned}$$

which is exactly the cochain level version of (28).

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