Graph coloring
Exercise class problems - volume 3

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1. Prove that the Petersen graph is not planar by finding an iterated subdivision of $K_{3,3}$ or K_5 as a subgraph.
2.
   - Prove that any planar graph has a vertex of degree at most 5.
   - Use it to deduce the **6-color theorem**: Every planar graph is 6-colorable.
3. How many edges must be removed from the Petersen graph to make the result planar?
4. Prove that $\overline{Q_3}$ is not planar.
5. Prove that Q_4 is not planar.
6. Prove that $v - e + f = k + 1$ in a planar graph with k connected components.
7. Prove (without invoking any difficult theorems) that any triangle-free planar graph is 4-colorable.
8. Suppose G is planar. Show that we can assign two colors to the vertices of G so that there are no monochromatic triangles. (A monochromatic triangle is a triangle in G with all vertices of the same color) You may use the 4-color theorem.

A graph is called k-critical if $\chi(G) = k$ and $\chi(H) < k$ for every proper subgraph $H \subseteq G$.

1. Prove that a graph is 3-critical if and only if it is an odd cycle.
2. Show that a k-critical graph G has $\delta(G) \geq k - 1$.
3. Suppose G is k-critical and $e \in E(G)$. Show that every $(k-1)$-coloring of $G - e$ assigns different colors to the endpoints of e.
4. Prove that $C_5 + K_k$ is $(3 + k)$-critical.