Graph coloring
Exercise class problems - volume 7

In the last lecture we defined the "generalized cube" graph $Q_d(u)$ as follows:

- The vertices are all binary sequences $(x_1, \ldots, x_d) \in \{0,1\}^d$.
- There is an edge from $\overline{x} = (x_1, \ldots, x_d)$ to $\overline{y} = (y_1, \ldots, y_d)$ if and only if $\overline{x}$ and $\overline{y}$ differ in exactly u positions.

Show that the following are equivalent definitions of $Q_d(u)$:

1. The vertices are all vertices of Q_d. Two vertices are adjacent if their distance in Q_d is exactly u.
2. The vertices are all subsets of $\{1, \ldots, d\}$. Two subsets A, B are adjacent if $|A \triangle B| = u$, where $\triangle$ is the symmetric difference $A \triangle B = (A \cup B) \setminus (A \cap B)$
3. The vertices are all integers in $\{0, \ldots, 2^d - 1\}$. Two numbers x, y are adjacent if $x \oplus y$ has exactly u nonzero digits in base 2, where $\oplus$ is bitwise XOR.
4. The vertices are the vertices of the cube $[0,1]^d \subseteq \mathbb{R}^d$. Two of them are adjacent if their Euclidean distance is $\sqrt{u}$.

Implement your favorite definition in SAGE. Now check (and later prove) that

- If u is odd then $Q_d(u)$ is bipartite (hence $\chi(Q_d(u)) = 2$).
- If u is even then $Q_d(u)$ has at least two connected components.

If $Q_d(u)$ has two components, we denote any of them by $Q'_d(u)$.

- Compute $\chi(Q'_5(2))$ and show that $\chi(\mathbb{R}^5) \geq 8$ (this is one off the best known bound $\chi(\mathbb{R}^5) \geq 9$).
- Compute $\alpha(Q'_{10}(4))$ and show that $\chi(\mathbb{R}^{10}) \geq 26$ (this is the best known bound).