Graph coloring
Graded homework 1

Problem 1. (2pt) Find the chromatic number of the graph G defined in accompanying file `graph.sage`.

Problem 2. (2pt) Consider the following algorithm for vertex coloring. Find the largest independent set of vertices, and color them with color 1. Remove those vertices, find the largest independent set in the remaining graph and color it with color 2, and so on until there are no more vertices left to color. Prove that there are infinitely many graphs G for which this algorithm will use more than $\chi(G)$ colors.

Problem 3. (2pt) Prove that $\max\{\chi(G), \chi(\overline{G})\} \geq \sqrt{|V(G)|}$ for any graph G.

Problem 4. (2pt) Let G, H be two graphs. The *substitution of H into G*, denoted $G[H]$, is the graph obtained by replacing every vertex of G with a copy of H, and replacing every original edge of G with a complete bipartite graph between the corresponding copies of H. Formally $V(G[H]) = V(G) \times V(H)$ and $(u, v)(u', v') \in E(G[H])$ iff either $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$. Sage calls this operation `G.lexicographic_product(H)`.

Prove that

$$\omega(G)\chi(H) \leq \chi(G[H]) \leq \chi(G)\chi(H).$$

Find an example with $\chi(G[H]) < \chi(G)\chi(H)$.

Problem 5. (2pt) Let $\gcd(a, b)$ denote the greatest common divisor of a and b. Let $n = 20162016$. Define G as the graph with vertex set $\{1, \ldots, n\}$ where two numbers $1 \leq a < b \leq n$ are adjacent if and only if $\gcd(a, b) = 1$. Find the exact value of $\chi(G)$.

Hints: $\chi(G_1 + \cdots + G_k) = \sum_{i=1}^{k} \chi(G_i)$. All graphs are undirected, finite and simple.

Deadline: Friday week 9, 04/03/2016, 10:16 am.