Graph coloring
Graded homework 2

Problem 1. (2pt)

a) Prove or disprove: if the only complex roots of $P_G(t)$ are 0 and 1 then G is a forest.

b) How many non-isomorphic graphs have chromatic polynomial $t^2(t-1)^8$?

c) Find all non-isomorphic graphs with chromatic polynomial $t(t-1)^3(t-2)$.

Problem 2. (2pt) A vertex coloring of G will be called *brilliant* if (1) every two adjacent vertices have different colors and (2) every two vertices which have a common neighbour also have different colors. Let $\chi_b(G)$ be the minimal number of colors required for a brilliant coloring of a simple graph G, and let $P_b(G,t)$ be the number of brilliant colorings of G with colors $\{1,\dots,t\}$.

Find all graphs G with $\chi_b(G) \le 2$ and show that $P_b(G,t)$ is a polynomial in t for every graph G.

Problem 3. (2pt) Prove that $\chi_l(G) + \chi_l(\overline{G}) \le |V(G)| + 1$ for any graph G, where χ_l is the list chromatic number.

Problem 4. (2pt) (This is an experimental problem; I am not expecting any proofs.) Let $g(n)$ be the expected number of colors used by the greedy algorithm to color a random graph from $G(n, \frac{1}{2})$.

- Compute and plot an experimental approximation of $g(n)$ for a sequence of reasonably large values of n, for example $n = 100, 200, \dots, 2000$.
- Speculate about the asymptotic behaviour of $g(n)$ as $n \to \infty$. In particular, what do you think about $\lim_{n\to\infty} \frac{g(n)}{n/\log_2 n}$?
- Find information about the expected value of $\chi(G)$ for $G \in G(n, \frac{1}{2})$. How well does the greedy algorithm perform?

Problem 5. (2pt) Choose and solve *one* of these problems.

(5.1) Let G be a nonempty graph. Simplify the expression

$$\sum_I P(G - I, -1)$$

where the sum runs over all independent sets I in G (including the empty one) and, as always, $G - X$ denotes the subgraph of G induced by the vertex set $V(G) - X$.

Hint: More generally, consider $\sum_I P(G - I, t)$.

(5.2) A vertex v of a directed graph is called a *source* if all the edges incident to v are pointing out of v. Suppose G is a nonempty graph with n vertices. Prove that the number of acyclic orientations of G having exactly one source equals $n \cdot (-1)^{n-1} \cdot [t]P_G(t)$.

Hint: Find a deletion-contraction rule for $a(G, v_0) :=$ the number of acyclic orientations of G in which some prescribed vertex v_0 is the unique source.

Deadline: Friday week 11, 18/03/2016, 10:16am.