Graph coloring

Lecture notes, vol. 1A, Intro to Graph Theory

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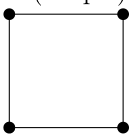
Formalities about the course:

- There will be 3 homework assignments. Each assignment can earn you 10 points. All in all, 30 points.
- You have the possibility to earn 3 points by volunteering to take one day of lecture notes and setting them up in LaTeX, for everyone to use. Please volunteer.
- You will have the possibility to present an open problem, related to Graph Coloring. This will be done in the exercise sessions. Presentations should be 5-10 min.
- There will be some computer experiments. We will use SageMath (`https://cloud.sagemath.com/`). Please set up an account befor exercise class on Friday, 12th of February.
- We will not be using Absalon, but you can find everything about the course on the course website (`http://www.mimuw.edu.pl/~aszek/chromatic/index.html`)
- For some literature; we will not follow a single book, at least not linearly. But Michal recommends the books by West, Bondy & Murty or Diestel. You can find the Diestel book online (go through the course website). The others should be available in the Library.

Introduction to Graph Theory.

Types of Graphs:

- A (simple) graph (we will mostly look at them)



- A multi-graph
  (multiple edges between a specific pair of vertices)
- A directed graph (the edges have a direction)
- A labelled graph (a labelled version of any other graph type. The edges have some more information)

Example 1. Real-world graphs. Roadmaps, tree of game, brain, chemistry (representation of chemical compounds, Systematic IUPAC), maps, scheduling (as a *matching problem* in a graph)

The First Problem in Graph Theory

Seven Bridges of Königsberg (now Kaliningrad, Russia)[1]. Euler[2] solved this problem in 1736 and is considered to be the first problem in Graph Theory.
The problem: *Is there a tour of the city which goes through each bridge once?* No!
Why not?

- A closed (begins and ends at the same point) walk visits each vertex an even number of times.
- Any walk (not necessarily closed) visits its non-endpoints an even number of times.

Definition 2. *A simple graph is a pair* $G = (V, E)$ *where*
V *is the set of vertices*
$E \subseteq \binom{V}{2}$*, is the set of edges. (A subset of the set of all 2-element subsets of* V*)*

Example 3. A graph G with vertices $V = \{a, b, c, d\}$ and edges $E = \{\{a, b\}, \{b, c\}, \{a, c\}, \{c, d\}\}$

Notation 4. *We will be using the following notations (obs. sometimes for simplicity symbols/letters are dropped here and there, but then it should be obvious what is meant):*

- $xy \in E(G)$ *instead of* $\{x, y\} \in E(G)$
- $N_G(x) = \{y : xy \in E(G)\}$ *is the neighborhood of* x *in* G. *(Example above:* $N(b) = \{a, c\}$*)*
- $xy \in E(G)$ *we say* x, y *are* <u>*adjacent*</u>*, or* <u>*neigbors*</u> *in* G.
- *a vertex is* <u>*incident*</u> *to the edges that contain it.*
- *UNLESS OTHERWISE NOTED, ALL GRAPHS ARE* <u>*FINITE AND SIMPLE*</u>

Example 5. Some key examples of families of graphs.

- <u>Cycles:</u> $C_n,\ n \geq 3$.
  $V = \{1, \ldots, n\}$
  $E = \{12, 23, \ldots, (n-1)n, n1\}$
- <u>Paths:</u> A subset of a cycle, $P_n,\ n \geq 1$
- <u>Complete graphs:</u> $K_n,\ n \geq 1$
  $V = \{1, \ldots, n\}$
  $E = \binom{V}{2}$, all possible pairs of V
  Note! $|E(K_n)| = \binom{n}{2} = \frac{n(n-1)}{2}$
- $\overline{K_n},\ n \geq 1$ (the complement of a complete graph)
  $V = \{1, \ldots, n\}$
  $E = \varnothing$
- <u>The Empty Graph:</u> denoted $\varnothing$;
  $V(\varnothing) = \varnothing$
  $E(\varnothing) = \varnothing$

Definition 6. *The complement of* G *is the graph* $\overline{G}$ *s.t.*

$$V(\overline{G}) = V(G)$$

$$E(\overline{G}) = \binom{V(G)}{2} \setminus E(G)$$

Example 7. $Q_n,\ n \geq 1$
$V(Q_n) = \{(x_1, \ldots, x_n) :\ x_i \in \{0, 1\}\}$, all binary sequences of length n.
$(x_1, \ldots x_n)(y_1, \ldots, y_n) \in E(Q_n)$ iff $(x_1, \ldots x_n)(y_1, \ldots, y_n)$ are different in exactly one position.

- $n = 1$
  $V(Q_1) = \{0, 1\}$
  $E(Q_1) = \{01\}$
- $n = 2$
  $V(Q_2) = \{00, 01, 10, 11\}$
  $E(Q_2) = \{\{00, 10\}, \{00, 01\}, \{01, 11\}, \{10, 11\}\}$
- $n = 3$
  $V(Q_3) = \{000, 001, 010, 100, 011, 101, 110, 111\}$
  $E(Q_3) = \{\{000, 001\}, \{000, 010\}, \ldots, \{011, 111\}\}$

Definition 8. $G = (V, E)$. *For a vertex* $v \in V$*, the* <u>*degree*</u> *of* v*,* $\deg(v)$*, is the number of edges incident to* v.

$$\deg(v) = |N(v)|$$

v is called <u>*isolated*</u> *if* $\deg(v) = 0$
v is called <u>*a leaf*</u> *if* $\deg(v) = 1$

Lemma 9.

$$\sum_{v \in V(G)} \deg(v) = 2 \cdot |E(G)|$$

Proof. Obvious, the sum counts every edge twice.
A Very Formal Proof: Let M be a matrix of size $|V| \times |E|$

$$M_{v,e} = \begin{cases} 1, & \text{if } v \text{ is incident to } e \\ 0, & \text{otherwise} \end{cases}$$

This is also called the incidence matrix of G. Let's see an example before we continue with the proof.

Example 10. $V = \{a, b, c, d\}$ and $E = \{ab, bc, cd\}$.

$$M = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccc} ab & bc & cd \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

Now, let's continue with the proof.
The number of $1's$ in M is:

$$\begin{cases} 2 \times |E|, & \text{(two } 1's \text{ in each column)} \\ \sum_v \deg(v), & (\deg(v)\, 1's \text{ in } n\text{-th row}) \end{cases}$$

(this is called double counting). □

Corollary 11. *The number of vertices of odd degree in every graph is even.*

Notation 12.

- $\Delta(G) =$ *maximum vertex degree in* G
- $\delta(G) =$ *minimal vertex degree in* G
- G *is called* d*-regular if* $\Delta(G) = \delta(G) = d$*, equivalently* $\forall v \in V$ $\deg(v) = d$*. (Example:* Q_n*)*

The Category of Graphs.

Definition 13. *A graph,* H*, is a subgraph of* G *if:*

- $V(H) \subseteq V(G)$ *and* $E(H) \subseteq E(G)$*. (*H *is also a graph, so* $E(H) \subseteq \binom{V(H)}{2}$*)*
- $W \subseteq V(G)$ *then* $G[W]$ *is the subgraph of* G *induced by* W*, which means,*

$$V(G[W]) = W$$

$$E(G[W]) = \{xy; xy \in E(G), x, y \in W\}$$

Example 14. Induced subgraphs

Definition 15. $f : G \to H$ *is a graph homomorphism if* $f : V(G) \to V(H)$*,*
s.t. $xy \in E(G) \Rightarrow f(x)f(y) \in E(H)$

Definition 16. G *and* H *are isomorphic if* $\exists\, f : G \to H$ *and* $g : H \to G$ *s.t.* $gf = 1_G, fg = 1_H$*.*

Observation 17. $1_G : G \to G$ *is homomorphism.* $f : G \to H$*,* $g : H \to K$ *are homomorphisms, then so is* $gf : G \to K$*.*

Remark 18. Deciding if two graphs are isomorphic is computationally hard. Proving non-isomorphism is usually about finding some invariant that distinguishes the two graphs.

References

[1] Seven Bridges of Königsberg, https://en.wikipedia.org/wiki/Seven_Bridges_of_K%C3%B6nigsberg

[2] Leonard Euler (1707-1783), https://en.wikipedia.org/wiki/Leonhard_Euler