Graph coloring

Lecture notes, vol. 10

Hall's marriage theorem and Edge coloring vs. 4-color theorem.

Lecturer: Michal Adamaszek Student: Hugrún Fjóla Hafsteinsdóttir

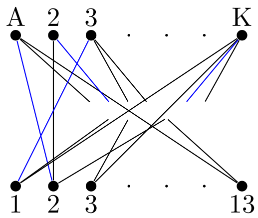
---

In the next pages, G is always a graph, $V(G)$ its set of vertices and $E(G)$ its set of edges.

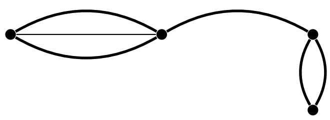
Hall's marriage theorem

G-bipartite graph with parts A, B. Suppose for any $X \subseteq A$, $|N(X)| \geq |X|$ where $N(X) = \bigcup_{x \in X} N_G(x)$. Then G has a matching of size $|A|$.

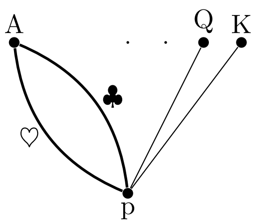
"Application": Split 52 cards into 13 piles of 4 cards each. Then we can choose 1 card from each pile, so that we have one card of each rank: $A, 2, 3, \ldots, K$. Construct a bipartite graph G, $V(G) = R \cup P$ (R — ranks and P — piles). Edges are $rp \in E(G)$ if pile p has a card of rank r. A matching in G with 13 edges determines a bijection $P \longleftrightarrow R$



How does this connect to Hall's Theorem? Take any subset $X \subseteq R$. X represents $4 \cdot |X|$ actual cards. These cards occupy $\geq |X|$ piles. This is exactly the statement $|N(X)| \geq |X|$, so Hall's theorem applies, and we have a matching of size 13. It's convenient now to use a *multigraph*, where many edges are allowed between two vertices



In our example I could take a bipartite multigraph G with one edge for every physical card



In this multigraph every vertex has a degree 4.

Theorem 1. *If G is a d-regular bipartite multigraph, then $\chi'(G) = d$.*

Proof. $V(G) = A \cup B$ - parts of the bipartition. $|A| = |B| = n$, because $|E(G)| = d \cdot |A| = d \cdot |B|$.

Induction on d:

- $d = 1$: G itself is a matching
- $d \geq 2$: Take any $X \subseteq A$ and let e_X be the number of edges in $G[X, N(X)]$

  $d \cdot |X| = e_X \leq d \cdot |N(X)|$, so $|X| \leq |N(X)|$.

  By Hall's Theorem, G has a matching of size n, $M \subseteq E(G)$. Since $G - M$ is a $(d-1)$-regular multigraph, by induction $\chi'(G) \leq 1 + (d-1) = d$.

□

Theorem 2. *(König) If G is a bipartite multigraph, then $\chi'(G) = \Delta(G)$.*

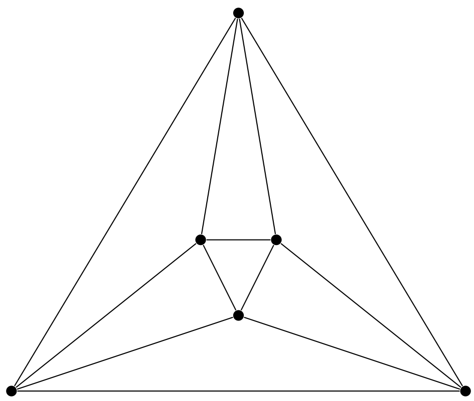
Proof. $V(G) = A \cup B$. We can assume $|A| = |B| = n$. (If not, then add extra isolated vertices to the smaller part). Write $\Delta := \Delta(G)$. If A has a vertex v of degree $< \Delta$, then also B has some vertex of degree $< \Delta$, call it u. (Because existence of $v \Rightarrow |E| < \Delta \cdot n$). Add a new edge uv to the graph. This process ends with Δ-regular bipartite H such that

$$V(H) = V(G),\ E(G) \subseteq E(H)$$

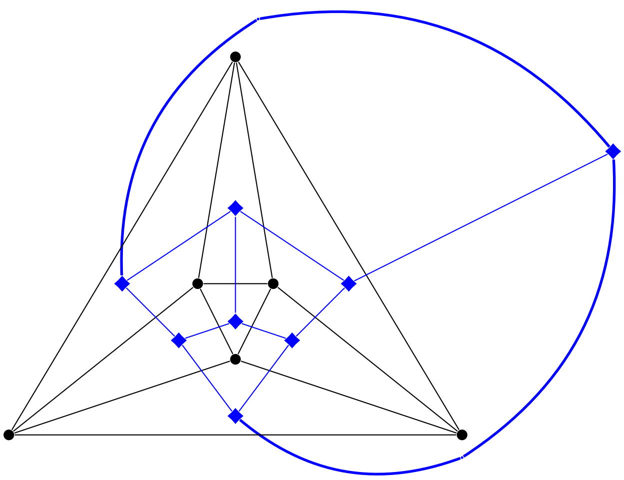
By previous theorem, $\chi'(G) \leq \chi'(H) = \Delta$ □

Edge coloring vs. the 4-color theorem.

Suppose G is a planar triangulation (a planar graph embedded so that all faces are triangles).



Definition 3. *The dual graph G^* of G is defined by the conditions: $V(G^*)$ = set of faces of the given embedding of G. $F_1F_2 \in E(G^*)$ if F_1, F_2 share a common edge. (We will usually represent each vertex of G^* as a point inside the corresponding face)*

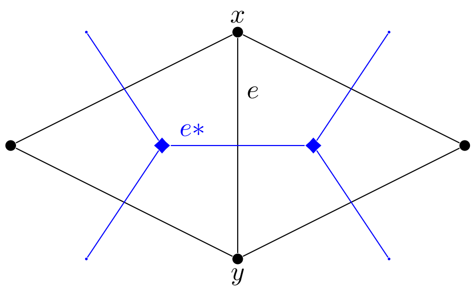


Observation 4. *By construction G^* has the following properties*

a) G^* is 3-regular. (Because every fave has 3 faces neighboring via a common edge)

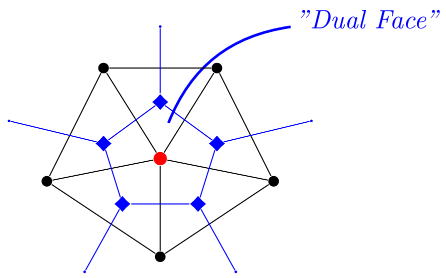
b) $|E(G^*)| = |E(G)|$

In fact every edge e of G determines an edge e^* of G^*.



c) G^* is planar

d) Every face of G^* contains exactly one vertex of G.



e) $|V(G^*)| = |F(G)|,\ |E(G^*)| = |E(G)|,\ |F(G^*)| = |V(G)|$.

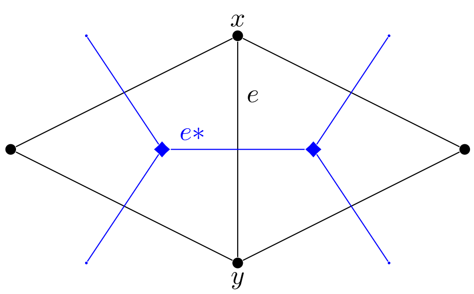
Theorem 5. (Tait '1878, Tait's attempt at the four-color problem).
Suppose G is a planar triangulation. TFAE:

a) G is 4-colorable

b) G^* is 3-colorable

Proof.
- *a) ⇒ b)*

Take a 4-coloring $c : V(G) \to \{00, 01, 10, 11\}$. If $e \in E(G)$, $e = xy$ then we color e^* with color $f(e^*) = c(x) \oplus c(y)$.



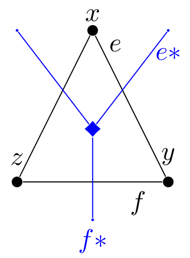
$\oplus$ is the coordinate-wise addition mod 2. (XOR = exclusive or)

$$0 \oplus 0 = 0 = 1 \oplus 1 \qquad 01 \oplus 11 = 10$$
$$1 \oplus 0 = 1 = 0 \oplus 1 \qquad A \oplus B = 00 \quad \text{iff } A = B$$

Let's check that f is an edge 3-coloring.

- $f(e^*) \neq 00$ because $c(x) \neq c(y)$, therefore $im(f) \subseteq \{01, 10, 11\}$

– Take $e^*, f^* \in E(G^*)$ sharing a common vertex



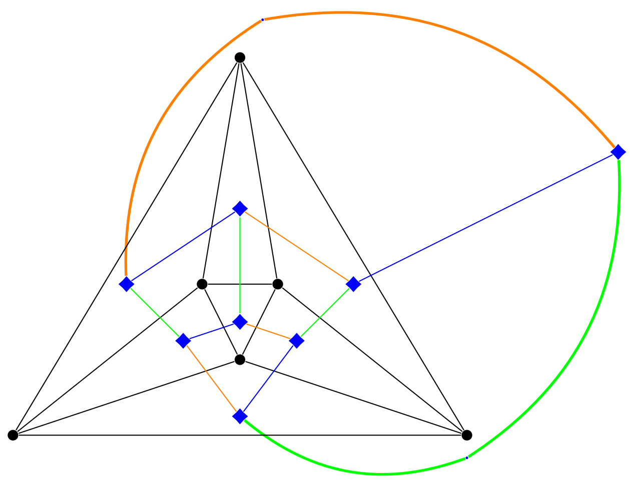
$$f(e^*) = c(x) \oplus c(y) \neq c(y) \oplus c(z) = f(f^*), \text{ because } c(x) \neq c(z)$$

- *b)* $\Rightarrow$ *a)*

  Start with a 3-coloring of $E(G^*)$:

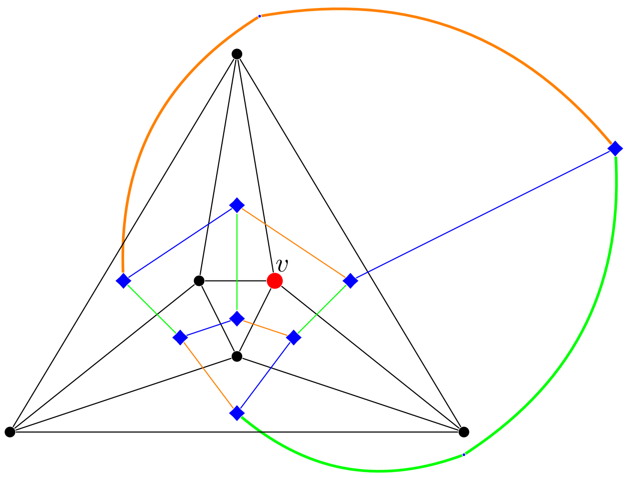
  $$f : E(G^*) \to \{1, 2, 3\}$$

  Since G^* is 3-regular, every color appears at every vertex of G^*. For $i = 1, 2$ let $H_i \subseteq G^*$ be the subgraph on the edges $f^{-1}(i) \cup f^{-1}(3)$.



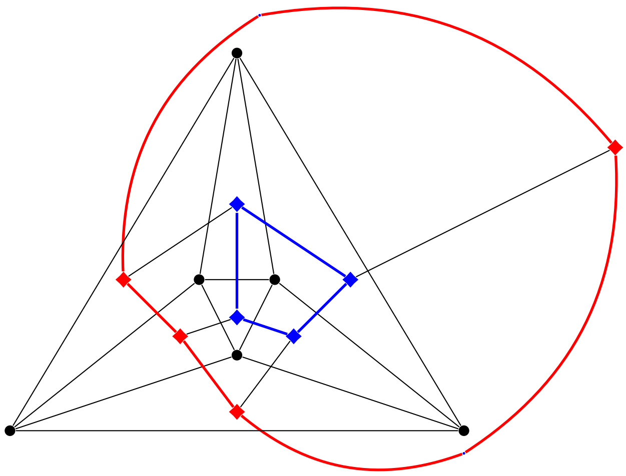
  H_i is 2-regular, hence it is a union of cycles. Construct a coloring $c : V(G) \to \{00, 01, 10, 11\}$ as follows:

  $$c(v) = x_1 x_2$$
  $$i = 1, 2 \qquad x_i = (\text{\# cycles in } H_i \text{ which contain } v \text{ inside}) \mod 2$$

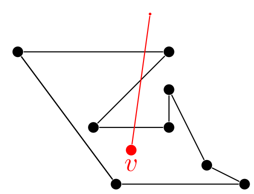


$$H_1 = f^{-1}(1) \cup f^{-1}(3), \; c(v) = (2 \mod 2)(1 \mod 2)$$
$$H_2 = f^{-1}(2) \cup f^{-1}(3)$$

The two cycles of H_1 that contain v inside are shown below, assuming color 1 is orange and color 3 is green:

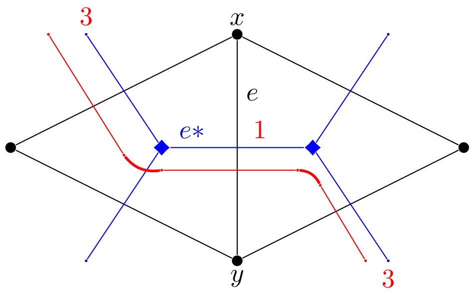


Remember: A cycle in H_i is a simple polygon in $\mathbb{R}^2$:



v is inside the polygon if a generic ray from v intersects the polygon an odd number of times.

This ends the example. Now we will prove that c is a vertex-coloring of G. Take $e = uv \in E(G)$. We want to show $c(u) \neq c(v)$. W.l.o.g. suppose that $f(e^*) = 1$



We know that e^* belongs to some cycle C of H_1

- v is inside C and u is outside C or vice versa.
- For any other cycle of H_1, both u, v are inside or both u, v are outside.

We can check both caims by counting (mod 2) the number of times a generic ray from u, v intersects a cycle of H_1.

By the claim $c(v)$ and $c(u)$ differ in x_1. Similarly:

If $f(e^*) = 2 \to c(v)$ and $c(u)$ differ in x_2

If $f(e^*) = 3 \to c(v)$ and $c(u)$ differ in x_1 and x_2

In any case $c(u) \neq c(v)$, so c is a 4-coloring

□