Graph coloring

Lecture notes, vol. 3, Bounds on $\chi(G)$ including Brooks Theorem

Lecturer: Michal Adamaszek Scribe: Rolf C. Jørgensen

Observation 1. *Every color class is an independent set.*

Proof. If $c(x) = c(y)$ then $xy \notin E(G)$. □

Lemma 2. *The following are equivalent:*

(1) G is k-colorable

(2) $V(G)$ can be partioned into k independent sets

(3) There exists a graph homomorphism $G \to K_k$

Proof. (1) $\Leftrightarrow$ (2): Assume (1) and choose a coloring $c\colon G \to \{1,\dots,k\}$. Then $c^{-1}(1),\dots,c^{-1}(k)$ partitions G into k independent sets. On the other hand if (2) holds we can write $V(G) = V_1 \cup \dots \cup V_k$ where each V_j is an independet set and $V_i \cap V_j = \emptyset$ when $i \neq j$. Define a coloring $c\colon G \to \{1,\dots,k\}$ by $c(v) = i$ if $v \in V_i$. Then c is clearly a coloring. So we have that (1) $\Leftrightarrow$ (2).

(2) $\Leftrightarrow$ (3): If $f\colon G \to K_k$ is a homomorphism then $f^{-1}(i)$ is an independent set. This shows (3) $\Rightarrow$ (2). Suppose $V(G) = V_1 \cup \dots \cup V_k$, $V_i \cap V_j = \emptyset$, with each V_i an independent set. Define $f\colon G \to K_k$ by $f(v) = i$ if $v \in V_i$. If $vw \in E(G)$ then $v \in V_i$ and $w \in V_j$ for $i \neq j$. Then $f(v)f(w) = ij \in E(K_k)$ since $i \neq j$. □

The next lemma says that "bigger graphs have bigger chromatic numbers".

Lemma 3. *If $H \subset G$ (H is a subgraph of G) then $\chi(H) \leq \chi(G)$.*

Proof. Any coloring $c\colon V(G) \to C$ restricts to a coloring $c\colon V(H) \to C$. □

Lower bounds on $\chi(G)$

Lemma 4. *For any graph G*

(1) $\chi(G) \geq \omega(G)$

(2) $\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$

Proof. (1): If $\omega(G) = k$, then G contains a clique on k vertices, i.e. $K_k \subset G$. Then $k = \chi(K_k) \leq \chi(G)$ by the lemma.

(2): Suppose $\chi(G) = k$. Then some color class has size at least $\frac{1}{k}|V(G)|$. Therefore we have an independent set of size $\geq \frac{1}{k}|V(G)|$, which means $\alpha(G) \geq \frac{1}{k}|V(G)|$. □

Example 5. $\chi(C_{2n+1}) = 3$ although $\omega(C_{2n+1}) = 2$ for $n \geq 2$. So it can happen that $\chi > \omega$.

Upper bounds on $\chi(G)$

Definition 6. The greedy coloring *relative to a vertex ordering $v_1, \dots v_n$ of the vertices of G is obtained by coloring in this order subject to the rule:*

$$c(v_i) = \text{ first available color that does not appear among } N(v_i) \cap \{v_1, \dots v_{i-1}\}$$

Theorem 7. *The greedy algorithm uses at most $\Delta(G) + 1$ colors. In particular $\chi(G) \leq \Delta(G) + 1$ for any G.*

Proof. When coloring v_i at most $\Delta(G)$ colors are forbidden, because v_i has at most $\Delta(G)$ neighbors, so it has $\leq \Delta(G)$ already colored neighbors. They use $\leq \Delta(G)$ different colors, so at least one color from $\{1, \dots, \Delta(G) + 1\}$ is available for v_i. □

Observation 8. *There is room for improvement if we choose some special vertex ordering.*

Example 9. Order the vertices so that $\deg(v_1) \geq \deg(v_2) \geq \cdots \geq \deg(v_n)$.

Theorem 10. *With respect to the above ordering, the greedy method uses at most* $1+\max_i(\min(\deg(v_i), i-1))$ *colors. In particular* $\chi(G) \leq 1 + \max_i \min(\deg(v_i), i-1)$.

Proof. v_i has at most $\min(\deg(v_i), i-1)$ neighbors among $v_1, \ldots, v_{i-1}$. ☐

Exercise 11. $\max_i \min(\deg(v_i), i-1) \leq \Delta(G)$ so this bound is at least as good as $\Delta(G)+1$ (often much better).

Example 12. Order the $v_i's$ so that for each i, v_i is some vertex of smallest degree in $G[\{v_1, \ldots, v_i\}]$ (the sub-graph induced by $\{v_1, \ldots, v_i\}$). Construction:

$$\begin{aligned} v_n &= \text{ vertex of minimal degree in } G \\ v_{n-1} &= \text{ vertex of minimal degree in } G - v_n \\ v_{n-2} &= \text{ vertex of minimal degree in } G - v_n - v_{n-1} \\ &\vdots \end{aligned}$$

Theorem 13. *With this ordering, the greedy method needs at most* $1 + \max_H \delta(H)$ *colors, where the maximum is over all induced subgraphs* H *of* G.

Proof. v_i has exactly $\deg_{G[\{v_1,\ldots,v_i\}]}(v_i)$ neighbors among $v_1, \ldots, v_{i-1}$. But that number is exactly $\delta(G[\{v_1, \ldots, v_i\}]) \leq \max_H \delta(H)$ ☐

Example 14. Where $\chi = \Delta + 1$:

(1) $\chi(K_n) = n = \Delta(K_n) + 1$

(2) $\chi(C_{2n+1}) = 3 = \Delta(C_{2n+1}) + 1$

Theorem 15 (Brooks)**.** *If* G *is a connected graph, which is not complete, nor an odd cycle, then* $\chi(G) \leq \Delta(G)$.

Proof. Let $\Delta = \Delta(G)$. If $\Delta \leq 2$ then G is a path or a cycle, and the result is easy. Assume $\Delta \geq 3$. Pick any vertex $v \in V(G)$ with $\deg(v) = \Delta$. Let $H = G - v$. The proof is by induction on the number of vertices $|V(G)|$. Suppose, for contradiction, G is not Δ-colorable. H is a graph with $\Delta(H) \leq \Delta$. By induction (since $|V(H)| = |V(G)| - 1$) H is $\Delta(H)$-colorable so in particular Δ-colorable. (Be careful: what if H is an odd cycle or a complete graph? We dealt with these cases in the exercises). Fix any coloring c of H with Δ colors. Since $\deg(v) = \Delta$ we can assume that all neighbors of V have different colors. Otherwise there is a spare color we can assign to v and G would be Δ-colorable. Label the vertices in $N(v)$ such that $N(v) = \{v_1, \ldots, v_\Delta\}$ and $c(v_i) = i$. Define $H_{i,j}$ be the subgraph of H induced by vertices of colors i and j ($i \neq j$, $i, j \in \{1, \ldots, \Delta\}$).

Claim (0)**.** $H_{i,j}$ *is bipartite and* $v_i, v_j \in V(H_{i,j})$.

Proof. Obvious. ☐

Claim (1)**.** v_i, v_j *are in the same connected component* $C_{i,j}$ *of* $H_{i,j}$

Proof. If not then $H_{i,j}$ is the disjoint union of two subgraphs: $H_{i,j} = H_{i,j}^{(1)} \cup H_{i,j}^{(1)}$ such that $v_i \in H_{i,j}^{(1)}$ and $v_j \in H_{i,j}^{(2)}$. Define a new coloring c' of H by swapping the colors in $H_{i,j}^{(2)}$ i.e. define $c'(v) = i$ if $c(v) = j$ and vice versa for all $v \in H_{i,j}^{(2)}$. (The new c' is still a coloring). With this new coloring v_i and v_j have the same colors, so we have a spare color for v which as before is a contradiction. ☐

Claim (2)**.** $C_{i,j}$ *is a path.*

Proof. First note that $\deg_{H_{i,j}}(v_i) = 1$ because otherwise v_i has two neighbors of color j. Moreover $\deg_H(v_i) \leq \Delta - 1$. Then there exist $k \neq i$ such that we can recolor v_i with color k and still have a coloring of H and we get a spare color for v. Analogously we have $\deg_{H_{i,j}}(v_j) = 1$. Let u be a degree ≥ 3 vertex in $C_{i,j}$ closest to v_i. Then $c(u) = i$ or $c(u) = j$, assume $c(u) = i$. Again: $\deg_H(u) \leq \Delta$, but neighbors of u have at most $\Delta - 2$ colors. We can recolor u with some $k \neq i, j$. Then v_i, v_j land in different components of $H_{i,j}$ contrary to claim 1. □

Claim (3)**.** *$C_{i,j} \cap C_{i,k} = \{v_i\}$ if $i \neq j \neq k \neq i$.*

Proof. If not then there is some $u \in C_{i,j} \cap C_{i,k}$ such that $u \neq v_i$. $\deg(u) \leq \Delta$ and u has two neighbors of color j and two of color k. Therefore neighbors of u use $\leq \Delta - 2$ colors and we can recolor u with some color $\neq i, j, k$ and get a proper coloring. The new coloring contradicts claim 1. □

If all of $v_1, \ldots, v_\Delta$ are pairwise adjacent then $G = K_{\Delta+1}$, or otherwise $\Delta(G) \geq \Delta + 1$. So, assume without loss of generality that $v_1v_2 \notin E(G)$. Consider the paths $C_{1,2}, C_{1,3}$. Let $C_{1,2} = v_1u \cdots v_2$. Flip the colors in $C_{1,3}$. We get a proper coloring of G. In this new coloring $C_{1,2}$ and $C_{3,2}$ both contain u contradicting claim 3. □

References

[1] West, *Introduction to graph theory*