Graph coloring

Lecture notes, vol. 4

Accuracy of lower bounds, probabilistic method and χ of some constructions

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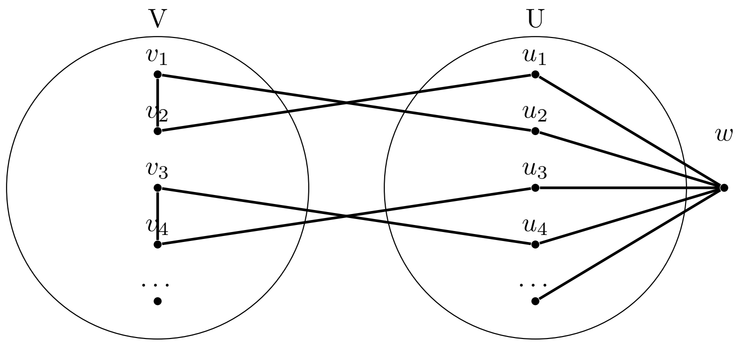
Accuracy of lower bounds

We know that $\chi(G) \geq \omega(G)$
If $G = C_{2n+1}$, $\quad n \geq 2$ then $\omega(G) = 2$, $\quad \chi(G) = 3$
Q: Is there a graph with $\omega(G) = 2$ and $\chi(G) = 4$?
A: Yes the smallest such is the Grötzsch graph on 11 vertices G_{11}
https://en.wikipedia.org/wiki/Grötzsch_graph

It is not possible to bound $\chi(G)$ in terms of $\omega(G)$.

Theorem 1. *For any $k \geq 2$ there is a triangle-free graph with $\chi(G) = k$*

Definition 2. *Suppose G is a graph with $V(G) = \{v_1, .., v_n\}$. The* Mycielski construction *$M(G)$ is a new graph with $V(M(G)) = \{v_1, .., v_n\} \cup \{u_1, .., u_n\} \cup \{w\}$, $\quad E(M(G)) = \{wu_i,\ i = 1, .., n\} \cup \{v_iv_j,\ u_iv_j\ :\ v_iv_j \in E(G)\}$.*



As an example have that $M(K_2) = C_5$ and $M(C_5) = G_{11}$.

Theorem 3. *(restated) If G is triangle-free and $\chi(G) = k$ then $M(G)$ is triangle free and $\chi(M(G)) = k + 1$.*

Proof.

1. $M(G)$ is still triangle free
   The only possibility is a triangle with 1 vertex from U and 2 vertices from V. However, by the definition of $M(G)$ we then have that $v_iv_jv_k$ is a triangle, contradicting that G was triangle-free.

2. $M(G)$ is $k + 1$ colorable
   We can color G with k colors (The set V in the picture). We use another color $(k + 1)$ for all vertices in U, and the last node w is colored with some color different from $(k + 1)$

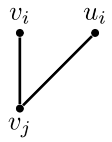
3. $M(G)$ is not k-colorable.
   Suppose otherwise, $c : V(M(G)) \to \{1, .., k\}$. Suppose wlog that w has color k, then U is colored with $k - 1$ colors. (Our goal is to show that we can color G with $k - 1$ colors)
   Let $A = \{v_i \in V,\ c(v_i) = k\}$.
   Recolor A by changing the color of each v_i to $c(u_i)$

$$c'(v_i) = \begin{cases} c(v_i) & \text{if } c(v_i) \neq k \\ c(u_i) & \text{if } c(v_i) = k \end{cases}$$

   C' uses only colors $\{1, .., k - 1\}$. Let's check that c' is a coloring of G. Suppose $c'(v_i) = c'(v_j)$

(a) If both of these $v_i, v_j \notin A$ then $v_iv_j \notin E(G)$

(b) If $v_i \in A$, $v_j \notin A$. Suppose $v_iv_j \in E(G)$, so that $u_iv_j \in E(M(G))$



Then $c(u_i) = c'(v_i) = c'(v_j) = c(v_j)$ Contradicting that c is a coloring of $M(G)$

(c) If $v_i, v_j \in A$. Then $c(v_i) = c(v_j) = k \Rightarrow v_iv_j \notin E(G)$

□

Recap: We can have triangle-free graphs with arbitrarily high χ. But is $M(G)$ just a special construction that achieves this? Not really. We will see that in the following:

Probabilistic method

Goal: Prove that objects with an interesting property P exists by showing that a random object has P with non-zero probability. For example: $P =$"triangle-free and $\chi(G) \geq k$"

Definition 4. *(construction). Fix $V = \{1, .., n\}$, $0 \leq p \leq 1$. Construct a graph on V by taking each edge ij, $0 \leq i < j \leq n$ with probability p, independently for each each pair.*
$G(n,p)$ is the probability space of all graphs obtained in this way.

A graph $G \in G(n,p)$ has probability

$$\mathbb{P}(G) = p^{|E(G)|}(1-p)^{\binom{n}{2}-|E(G)|}.$$

The expected number of edges/triangles in a random graph from $G(n,p)$ is

$$\mathbb{E}[\#\text{edges}] = \binom{n}{2}\mathbb{P}[ij \text{ is an edge}] = p\binom{n}{2}$$

$$\mathbb{E}[\#\text{of triangles in } G(n,p)] = \binom{n}{3}\mathbb{P}[ijk \text{ is a triangle}] = \binom{n}{3}p^3$$

Sage can generate random graphs from $G(n,p)$ (`graphs.randomGNP(n, p)`). For the next part we need a few prerequisites:

1. $1 - x \leq e^{-x}$,
2. $\binom{n}{k} \leq n^k$,
3. Markov's inequality: if X is a non-negative random variable, $t > 0$, then $\mathbb{P}[X > t] \leq \frac{1}{t}\mathbb{E}[X]$.

Theorem 5. *For every $k \geq 2$ there is a triangle-free graph with $\chi(G) \geq k$.*

Remark: Not only "there is" but "there are many".

Proof. Take $G \in G(n,p)$ where $p = \frac{1}{n^{5/6}}$. Let $X = \#$ of triangles in G.
$\mathbb{E}[X] = \binom{n}{3}p^3 \leq n^{1/2}$. By Markov's inequality $\mathbb{P}[X > 10n^{1/2}] \leq \frac{1}{10}$, which means that a typical G has very few triangles.
To show that $\chi(G)$ is "large" we will prove that $\alpha(G)$ is "small". Let $a = \frac{3}{p}\ln n$

$$\mathbb{P}[\alpha(G) \geq a] \leq \binom{n}{a}\mathbb{P}[\text{exists independent set of size } a] = \binom{n}{a}(1-p)^{\binom{a}{2}} \leq n^a e^{-p\binom{a}{2}} \to 0, \quad n \to \infty$$

where the last limit can be computed by plugging in the formulae for p and a in terms of n.

For sufficiently large n we have $\mathbb{P}[\alpha(G) \geq a] < \frac{1}{10}$. Then

$$\mathbb{P}[X < 10\sqrt{n} \text{ and } \alpha(G) < a] \geq \frac{8}{10}.$$

We showed, with probability $\geq \frac{8}{10}$, a random graph from $G(n,p)$, $p = \frac{1}{n^{5/6}}$, has $< 10\sqrt{n}$ triangles, and $\alpha(G) < a < 3n^{5/6} \ln n$

Completing the proof:

Take such a random G. Remove at most $10\sqrt{n}$ vertices so we get a triangle free graph G'

$|V(G')| \geq \frac{n}{2}$ (for large n) and

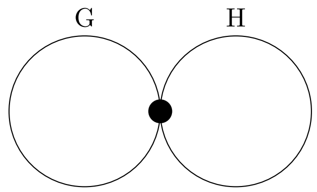
$$\chi(G') \geq \frac{|V(G')|}{\alpha(G')} \geq \frac{n/2}{3n^{5/6} \ln n} = \frac{1}{6} \frac{n^{1/6}}{\ln n} \to \infty, \quad n \to \infty$$

So $\chi(G')$ can be arbitrarily large □

χ of some constructions on graphs

1. Disjoint union $G \sqcup H$. $\chi(G \sqcup H) = \max(\chi(G), \chi(H))$. As $G \subseteq G \sqcup H \supseteq H$, we need at least enough colors to color H, G individually.

2. Wedge $G \vee H$ Graphs joined at a single vertex.



$\chi(G \vee H) = \max(\chi(G), \chi(H)$. After coloring G color H, perhaps permuting the colors so that they agree on the common vertex..

3. The sum (join) $G + H$. It is the disjoint union together with all edges between $V(G)$ and $V(H)$
$\chi(G + H) = \chi(G) + \chi(H)$. As the colors of G must be different from the colors of H.

4. The cartesian product $G \square H$
$V(G \square H) = V(G) \times V(H)$
$\{(u,v),(u',v')\}$ is an edge iff ($u = u'$ and $vv' \in E(H)$) or ($uu' \in E(G)$ and $v = v'$).

Lemma 6. $\chi(G \square H) = \max(\chi(G), \chi(H))$

Proof. $G \subseteq G \square H \supseteq H$ so $\chi(G \square H) \geq \max(\chi(G), \chi(H))$. Suppose G and H both have coloring with color set $\{0, .., k-1\}$ ($k = \max(\chi(G), \chi(H))$). Le these colorings be

$$f : V(G) \to \{0, .., k-1\},$$
$$f' : V(H) \to \{0, .., k-1\}.$$

Define $F(u,v) = f(u) + f'(v) \mod k$. We will check that F is a coloring. Let $(u,v)(u',v') \in E(G \square H)$. Wlog let $u = u'$ and $vv' \in E(H)$, then:

$$\begin{aligned} F(u,v) &= f(u) + f'(v) \mod k \\ F(u',v') &= f(u') + f'(v') \mod k \\ &= f(u) + f'(v') \mod k \\ &\neq f(u) + f'(v) \mod k \quad \text{because } vv' \in E(H) \\ &= F(u,v). \end{aligned}$$

□

As an exercise, show that $\chi(G) \geq h$ iff $\alpha(G \Box K_k) \geq |V(G)|$.