Graph coloring

Lecture notes, vol. 6, Lists, triangulation and the art gallery problem

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Theorem 1 (5-color theorem)**.** *If G is planar then $\chi(G) \leq 5$.*

Proof. Later: "Proof from the book" (Aigner, Zeigler). □

List coloring, examples:

1 A graph $G(V, E)$ with

- V = courses
- E = conflicts (student in both courses)

can be subject to a list-coloring should there restrictions to for instance the choices of days that a course can be held at.

2 Soduku $\equiv$ coloring of 9×9 grid with restrictions in the form of already given numbers.

As a definition:

Definition 2. *Let G be any graph, for every vertex $x \in V(G)$ we have a set of colors $L(x)$ (of colors available to x).*

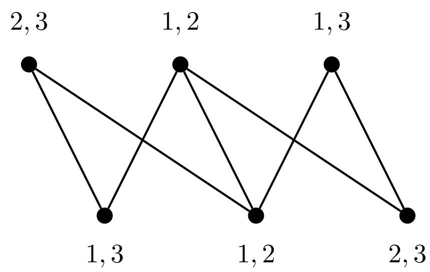
A L-coloring of G is a coloring $c : V(G) \to \bigcup_x L(x)$ such that $c(x) \in L(x)$ for all x.

The list-chromatic number $\chi_\ell(G)$ is the smallest k s.t. G has a L-coloring for any choice of list satisfying $|L(x)| \geq k$.

G is k-list-colorable (k-choosable) when $\chi_\ell(G) \leq k$.

Example 3. Here are some various examples and facts:

- $L(x) = \{1, \dots, k\}$ for all $x \in V(G)$ then L-coloring $\equiv$ k-coloring in the normal sense
- $\chi(G) \leq \chi_\ell(G)$
- Example with $\chi(G) < \chi_\ell(G)$:



For each x, $|L(x)| = 2$. But for these list there are no L-coloring. Therefore $\chi_\ell(G) \geq 3$, while $\chi(G) = 2$ since the graph is bipartite.

- $\chi_\ell(G) \leq \Delta G + 1$, same proof as the non-list chromatic number.
- Brooks Theorem holds.

Question: Can we construct G with small $\chi(G)$ and large $\chi_\ell(G)$?

Proposition 4. *For every $k \geq 2$ there exist a graph with $\chi(G) = 2$ and $\chi_\ell(G) > k$*

Proof. Take $A = B = \binom{\{1,\ldots,2k-1\}}{k}$, (the set of all k-subsets of $\{1,\ldots,2k-1\}$).

Let $G = K_{A,B}$ be the complete bipartite graph with parts A and B.

Example 5. Take $k = 3, |A| = |B| = \binom{5}{3} = 10$

That is A and B will consist of 10 vertices each with a list of three numbers made of the various permutations of $[1,2,3,4,5]$ and every vertex in A is connected to every vertex in B. So $|V| = 20, |E| = 100$.

(*proof* cont.) For $X \in A$ or $X \in B$ set $L(X) = X$ and note $|L(X)| = k$. We claim that $K_{A,B}$ has no L-coloring.

Suppose c is an L-coloring. Then $c(A) \subseteq \{1,\ldots,2k-1\}$, $|c(A)| \geq k$. $\leftarrow$ will be true, or otherwise $|c(A)| \leq k-1$ and then there is a $X \subseteq \{1,\ldots,2k-1\}$ such that $|X| = k$ and $X \cap c(A) = \emptyset$. Then X cannot be colored with a color from $L(X)$.

Similar for B, that is $c(B) \subseteq \{1,\ldots,2k-1\}$, $|c(B)| \geq k$.

It follows that $c(A) \cap c(B) \neq \emptyset$ so two vertices on opposite sides have the same color. That means $\chi_\ell(K_{A,B}) \geq k+1$, but $\chi(K_{A,B}) = 2$. □

Theorem 6. *(Thomassen 1994)*

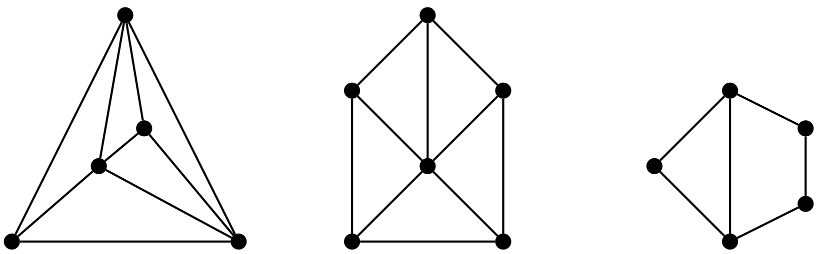
Every planar graph satisfies $\chi_\ell(G) \leq 5$. *(It is 5-list-colorable).*

Remark 7. This is stronger than the 5-color theorem, because $\chi(G) \leq \chi_\ell(G)$.

Definition 8. *An embedding of a planar graph G is called:*

- *A triangulation if every face is a triangle (also the unbounded one)*
- *A near-triangulation if every bounded face is a triangle and the unbounded one is a cycle.*

Example 9. Various examples of Definition 8.



From left to right: A triangulation, a near triangulation and the final picture is neither.

Remark 10. Every embedding can be extended to a triangulation on the same vertex set. (By only adding edges)

Proof. Add diagonals as needed. It also work for the unbounded face. □

In other words: For any planar G there is a triangulation H such that

$$V(G) = V(H),\ E(G) \subseteq E(H).$$

In particular

$$\chi(G) \leq \chi(H)$$
$$\chi_\ell(G) \leq \chi_\ell(H)$$

Proposition 11. *Suppose G is near-triangular with outer cycle $\mathbb{O} = x_1,\ldots,x_k$ and assume the following exist:*

- $L(x_1) = \{a\}$, $L(x_2) = \{b\}$, $a \neq b$
- $|L(x_i)| \geq 3$ *for all* $i = 3,\ldots,k-1$

- $|L(y)| \geq 5$ *for all* $y \notin \mathbb{O}$.

Then G *is* L*-colorable.*

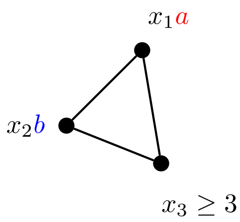
Remark 12. This proposition implies Thomassens theorem as follows:

Take H, any planar graph, with lists L and $|L(x)| \geq 5$. Extend H to a triangulation $H \subseteq G$ (in particular, G is a near-triangulation). Choose any x_1, x_2 on the outer face. Restrict $L(x_1)$ and $L(x_2)$ to one element and voila! $\rightarrow$ the proposition applies to G.

L-coloring of G gives an L-coloring for H.

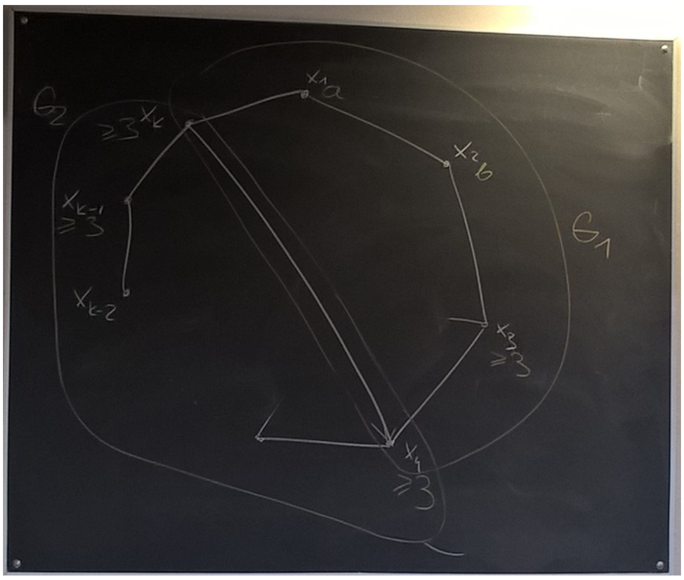
Proof. Proof of proposition 11:

- $|V(G)| = 3$:



Then we will have a spare color for x_3.

Case 1:
There is an edge $x_k x_j$, $j = 2, \ldots, k-2$



G_1 = graph bounded by $x_1 x_2 \ldots x_j x_k \rightarrow L$-color G_1 by induction.

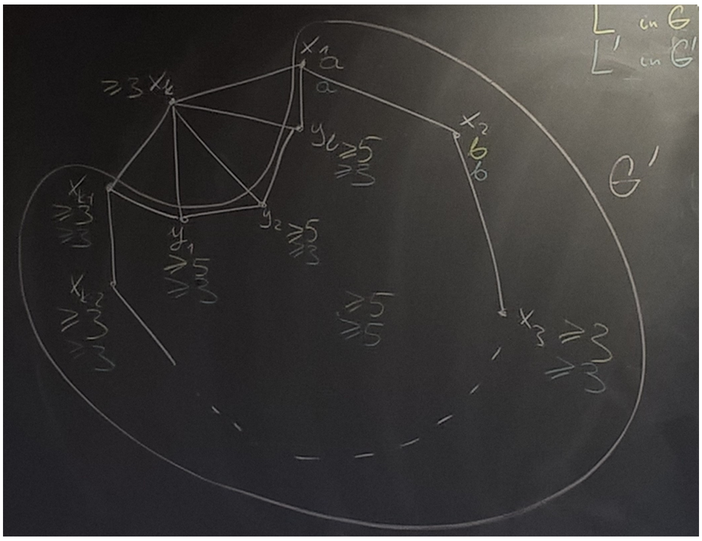
G_2 = graph bounded by $x_k x_j \ldots x_{k-1} \rightarrow L$-color G_2 by induction.

$\rightarrow$ L-coloring of G. Note that after coloring G_1 some colors are not available for certain vertices of G_2, but there is enough left just to use the induction hypothesis for G_2.

Case 2:
There are no edge $x_k x_j$, $j = 2, \ldots, k-2$.

- Around x_k we must have a sequence of triangles, since the interior is triangulated.

- Let $N(X_k) = \{x_1, x_{k-1}, y_1, \dots, y_\ell\}$
- Pick $c, d \in L(X_k)$, $c \neq d$, $c, d \neq a$.
- Set $G' = G - x_k$ and note that G' is a near-triangulation.



Consider the list:

$$L'(y_i) = L(y_i) \backslash \{c, d\} \ , \ L'(x) = L(x) \text{ for any other } x \in G'$$

$\rightsquigarrow$ There is a L'-coloring of G' by induction.

$$c(y_i) \notin \{c, d\} \ , \ c(x_1) \notin \{c, d\}$$

$\rightsquigarrow$ color x_k with either c or d depending on $c(x_{k-1})$. □

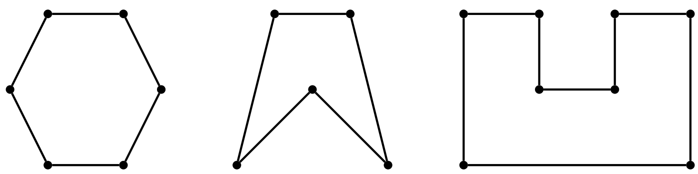
Remark 13. There are planar not-4-list-colorable graphs. (Voigt '93 example with ≈ 300 vertices).

Remark 14. Did not use Euler, upper/lower bounds. Only geometric properties.

Application: The art gallery problem

Suppose P is a polygon in $\mathbb{R}^2$ with n vertices.

Example 15. :

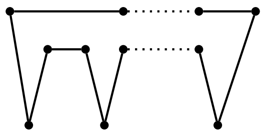


We assume the bounded region is the floor plan af an art gallery.

How many guards are needed to guard each point in sight?

Problem 16.
- *Find a gallery with 6 vertices requiring ≥ 2 guards.*
- *Find a gallery with as few vertices as possible requiring $\geq k$ guards.*

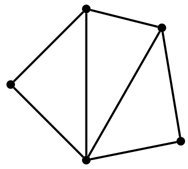
Observation 17. *There are n-vertex galleries requiring at least $\lfloor \frac{n}{3} \rfloor$ guards.*



Theorem 18. *Rephrasing the art gallery problem:*
Every n-vertex gallery can be guarded by $\lfloor \frac{n}{3} \rfloor$ *guards. (≈ 70's Chvatal).*

Definition 19. *A planar embedding is called a* polygon triangulation *if it is a near-triangulation and all vertices lie on the outer cycle.*

Example 20. "Polygon triangulated by diagonals"



Observation 21. *A triangulated polygon with n vertices has* $2n-3$ *edges, n on the outer cycle and* $n-3$ *diagonals.*

Observation 22. *Every triangulated polygon have a vertex of degree 2.*

Proof. The shortest diagonal cuts off such a vertex. □

Observation 23. *Every triangulated polygon is 3-colorable.*

Proof. Take v to be a vertex of degree 2. $G - v$ is a triangulated polygon. Color $G - v$ with 3 colors, the neighbours of v will use 2 colors. Color v with the spare color. □

Proof. Proof of Theorem 18:

- Let G be planar polygon with n vertices.
- First triangulate using diagonals (Exercise: show that this is always possible).
- The resulting triangular polygon is 3-colorable.
- Some color class will have $\leq \lfloor \frac{n}{3} \rfloor$ elements
- Place guards at the vertices of that color.

□