Graph coloring

Lecture notes, vol. 7,
The chromatic polynomial

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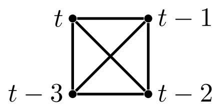
Further on planar graphs. In the 1930s Birkhoff and Whitney had the idea to, instead of constructing one 4-colouring for a planar graph, rather count all 4-colourings, and determine whether the result is > 0. This leads to the notion of the chromatic polynomial.

Definition 1. *For a graph G we define the chromatic function $P_G(t)$, $P(G,t)$ or $P(t)$, as*

$$P_G(t) = \#\text{ of vertex colourings of } G \text{ with colours } \{1,\ldots,t\}$$
$$= |\{c : V(G) \to \{1,\ldots,t\} : c \text{ is a colouring}\}|$$

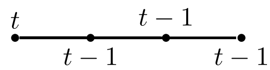
Example 2.

- $G = \overline{K_n}$, $P_G(t) = t^n$.
- $G = K_n$, $P_G(t) = t(t-1)\cdots(t-(n-1))$.

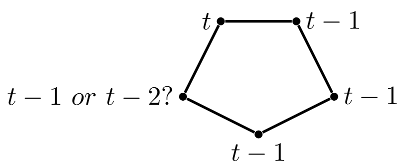


Notation 3. *$t^{\underline{n}}$ is the n'th falling factorial of t. $P_G(k) = 0$ when $k = 1,\ldots,n-1$.*

- $G = P_n$, $P_G(t) = t(t-1)^{n-1}$.



- $G = \emptyset$, $P_G(t) = 1$.
- We find a complication with cycles. $G = C_5$



Our computation doesn't keep track of whether 1 and 4 are coloured differently.

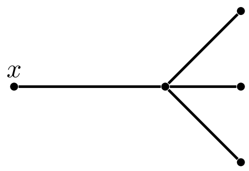
- $\chi(G) = \min\{k \in \mathbb{N} : P_G(k) > 0\}$, and
- The 4-colour theorem $\equiv P_G(4) > 0$ for a planar graph G.

Lemma 4. *If G is a tree, then*

$$P_G(t) = t(t-1)^{n-1}$$

where $n = |V(G)|$.

Proof. If G is a single vertex, then $P_G(t) = t = t(t-1)^{1-1}$. Pick a leaf $x \in V(G)$



$$P_G(t) = P_{G-x}(t)(t-1) = t(t-1)^{n-2}(t-1) = t(t-1)^{n-1}$$

by induction. □

Lemma 5. *For the disjoint union of graphs G, H, $P(G \sqcup H, t) = P(G,t)P(H,t)$*

Proof. Any pair (colouring of G, colouring of H) gives a colouring of $G \sqcup H$. □

The chromatic function is indeed the chromatic polynomial.

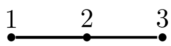
Proposition 6. *For a graph $G \neq \emptyset$, let $\pi_i(G)$ be the number of ways to partition $V(G)$ into exactly i non-empty independent sets. Then*

$$P_G(t) = \sum_{i=0}^{n} \pi_i(G) t^{\underline{i}} \quad , \qquad n = |V(G)|$$

Proof. The choice of a colouring with colours from $\{1, \dots, t\}$ is the same as

- partitioning into i independent sets: $\pi_i(G)$ ways to do this.
- colouring each part with a different colour: $t(t-1)\cdots(t-(i-1)) = t^{\underline{i}}$ ways.

And we do this for $1 \leq i \leq n$ □

Example 7. Let $G = P_3$. 

$$\begin{aligned} \pi_1 &= 0 \\ \pi_2 &= 1 \quad , \qquad 1-2-1 \\ \pi_3 &= 1 \quad , \qquad 1-2-3 \\ \pi_{\geq 4} &= 0 \end{aligned}$$

Then

$$P_{P_3}(t) = \pi_2 t^{\underline{2}} + \pi_3 t^{\underline{3}} = t(t-1) + t(t-1)(t-2) = t(t-1)^2$$

Remark 8. Hence from now on we call $P_G(t)$ the chromatic polynomial.

Proposition 9. *$G = (V, E)$ is a graph. If $G \neq \emptyset$, then*

$$P_G(t) = \sum_{F \subseteq E} (-1)^{|F|} t^{c(F)}$$

where $c(F)$ is the number of connected components of (V, F).

Before the proof we remind ourselves of the Inclusion-Exclusion principle.

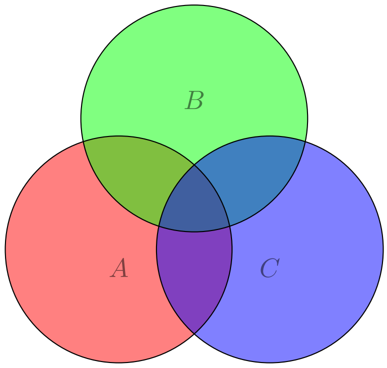
Fact 10. *Suppose A is a set, and $A_1, \dots, A_k$ are subsets of A. For any $X \subseteq \{1, \dots, k\}$ define*

$$A_X = \bigcap_{i \in X} A_i$$

Then

$$\left| \bigcup_{i=1}^{k} A_i \right| = \sum_{X \subseteq \{1,\dots,k\}} (-1)^{|X|} |A_X|$$

Example 11. Assume we have some ambient set $\mathcal{A}$, with $A, B, C \subset \mathcal{A}$.



Then we have

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$

$$|\overline{A \cup B \cup C}| = |\mathcal{A}| - |A| - |B| - |C| + |A \cap B| + |B \cap C| + |A \cap C| - |A \cap B \cap C|$$

Proof of proposition 9. Define $A = \{g : V \to \{1, \dots, t\}\}$ - all functions, not just colourings. For every $e = xy \in E$, let $A_e = \{g \in A : g(x) = g(y)\}$. Now for $F \subseteq E$, let $A_F = \bigcap_{e \in F} A_e$. Clearly $|A_F| = t^{c(F)}$, because $g \in A_F$ must be constant on every component of (V, F). But then we're done, as

$$P_G(t) = \left| \overline{\bigcup_{e \in E} A_e} \right| \overset{Incl.-excl.}{=} \sum_{F \subseteq E} (-1)^{|F|} |A_F|$$

□

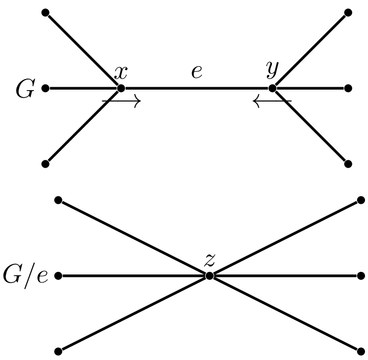
Notation 12. *If $P(t)$ is a polynomial, we write $[t^k]P(t)$ for the coefficient of t^k in $P(t)$.*

Lemma 13. *If G is a graph with n vertices and m edges, then $P_G(t)$ is a polynomial of degree n, with*

$$[t^n]P_G(t) = 1 \quad , \qquad [t^{n-1}]P_G(t) = -m$$

Proof. By proposition 9, we have $c(F) \le n \Rightarrow \deg P_G \le n$. Now then $c(F) = n$ iff. $F = \emptyset$, which implies t^n appears exactly once in $P_G(t)$ with coefficient $(-1)^{|\emptyset|} = 1$.
$c(F) = n - 1$ iff. $F = \{e\}$, which implies t^{n-1} appears with coefficient $(-1)^{-1}m = -m$. □

Notation 14. *For $e \in E(G)$ we write: $G - e$ for G with e removed, and G/e for G with e contracted.*



Proposition 15. *(Deletion-contraction rule). If $e \in E(G)$, then*

$$P_G(t) = P_{G-e}(t) - P_{G/e}(t)$$

Proof. Let $e = xy \in E(G)$.

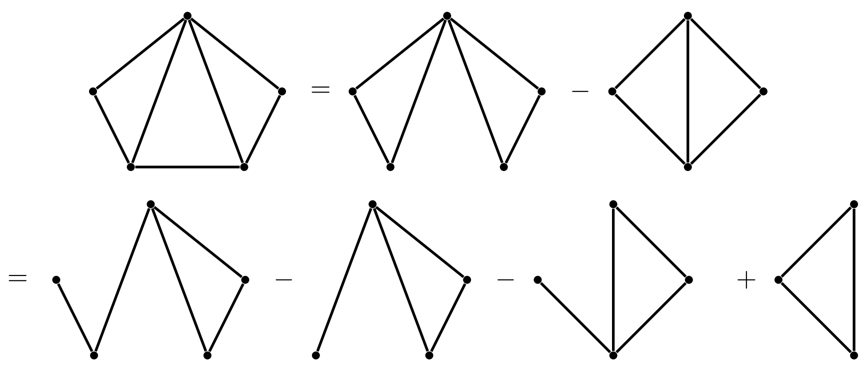
$$\begin{aligned} P_{G-e}(t) &= \#\text{ colourings with } c(x) \neq c(y) + \#\text{ colourings with } c(x) = c(y) \\ &= P_G(t) + P_{G/e}(t) \end{aligned}$$

□

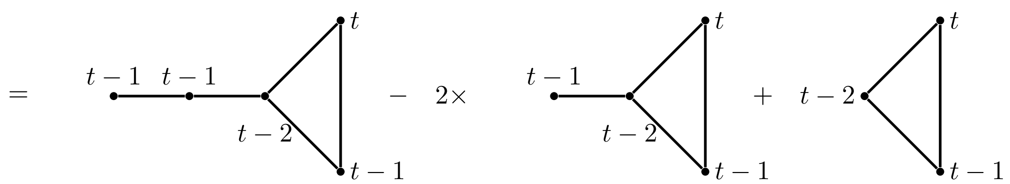
Remark 16. $|E(G-e)| = |E(G)| - 1$ and $|V(G/e)| = |V(G)| - 1$, so we could define $P_G(t)$ recursively

$$P_G(t) = \begin{cases} t^{|V(G)|} & \text{if } G \text{ has no edges} \\ P_{G-e}(t) - P_{G/e}(t) & \text{if } e \in E(G) \end{cases}$$

Example 17. G is a graph on 5 vertices, and we run the above.



Now arrange the graphs accordingly s.t.



$$= t(t-1)^3 - 2t(t-1)^2(t-2)(t-2) + t(t-1)(t-2)$$

Example 18. $P(C_n, t) = P(P_n, t) - P(C_{n-1}, t)$. We have

$$\begin{aligned} P(P_n, t) &= t(t-1)^{n-1} \\ P(C_3, t) &= t(t-1)(t-2) \end{aligned}$$

By induction $P(C_n) = (t-1)^n + (-1)^n(t-1)$. We can check that $P(C_n, 2) \neq 0$ iff. $2|n$.

$$P(C_n, 2) = 1 + (-1)^2 = \begin{cases} 2, & 2 \mid n \\ 0, & 2 \nmid n \end{cases}$$

Proposition 19. *Let $G \neq \emptyset$ be a graph with n vertices, m edges and c connected components. The coefficients of $P_G(t)$ alternate in signs, i.e.*

$$P_G(t) = \sum_{i=0}^{n} (-1)^i c_i(G) t^{n-i}$$

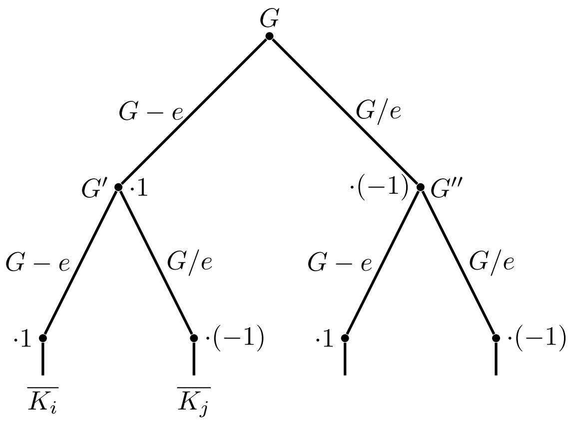
where $c_i(G) \geq 0$. Moreover $c_i(G) = 0$ for $i > n - c$ and $c_{n-c}(G) \neq 0$.

Simply put:

$$P_G(t) = t^n - mt^{n-1} + c_2(G)t^{n-2} - \cdots + (-1)^{n-c} c_{n-c}(G) t^c$$

the last term t^c is with t to the power of the number of connected components. Exercise: do a proof by induction.

Proof. Deleting keeps the sign, and contracting changes the sign.



Every branch of the deletion-contraction tree ends with some $\overline{K_i}$, $1 \leq i \leq n$. Every branch ending with $\overline{K_i}$ contributes

$$(-1)^{n-i} t^i$$

because the path from G to $\overline{K_i}$ contains $n-i$ contractions, i.e. $n-i$ sign changes. The proposition holds with

$$c_i(G) = \# \text{ of branches ending with } \overline{K_i}$$

and clearly $c_i(G) \geq 0$.

To prove that $c_i(G) = 0$ for $i > n - c$ note no branch of the tree ends with with a graph on less than c vertices. Moreover, there is at least one branch with ends exactly with K_c (apply contractions all the time), so $c_{n-c}(G) > 0$. □