Graph coloring

Lecture notes, vol. 8

Chromatic polynomials, orientations, chromatic roots

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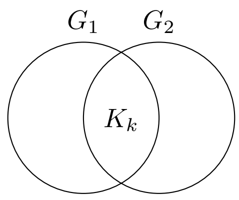
In the next pages, G is always a graph, $V(G)$ its set of vertices and $E(G)$ its set of edges.

Lemma 1. *Let G, G_1, G_2 be graphs such that $G = G_1 \cup G_2$ and $G_1 \cap G_2 \simeq K_k$ for some $k \geqslant 0$. Then*

$$P_G(t) = \frac{1}{t^{\underline{k}}} P_{G_1}(t) P_{G_2}(t).$$

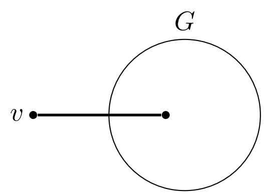


Proof. Colour G_1 and colour G_2. Since $G_1 \cap G_2 \simeq K_k$, $G_1 \cap G_2$ uses k different colours. It means that the colourings of G_1 and G_2 agree in $\frac{1}{P_{K_k}(t)}$ fraction of pairs. □

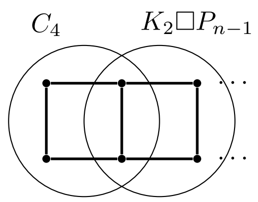
Application 2.

1. $G = G_1 \sqcup G_2$ ($k = 0$), then
$$P_G(t) = P_{G_1}(t) P_{G_2}(t),$$

2. v is a leaf in G ($k = 1$), then
$$P_G(t) = \frac{1}{t} P_{K_2}(t) P_{G-v}(t) = \frac{1}{t} t(t-1) P_{G-v}(t) = (t-1) P_{G-v}(t),$$



3. $G = K_2 \square P_n = C_4 \cup K_2 \square P_{n-1}$ ($k = 2$), then
$$P_G(t) = \frac{1}{t(t-1)} P_{C_4}(t) P_{K_2 \square P_{n-1}}(t),$$
and we can use this method recursively.



Summary. G is a graph with chromatic polynomial $P_G(t)$.

- $n = |V(G)| = deg(P_G)$,
- $m = |E(G)| = -[t^{n-1}]P_G(t)$,
- The number of connected components is $= max\{c : \; t^c \mid P_G(t)\}$,
- $\chi(G) = 1 + max\{k : \; (t-k) \mid P_G(t)\} = 1 + max\{k : \; t^{\underline{k}} \mid P_G(t)\}$,
- The number of triangles is $= \binom{m}{2} - [t^{n-2}]P_G(t)$ (will be proved during the next exercise session),
- The coefficients of the polynomial are integers with alternating signs.

Remark 3. It is hard to computer $P_G(t)$, otherwise we could easily compute $\chi(G)$. It is also hard to recognize chromatic polynomials.

Theorem 4. (June Huh, 2010) *Suppose G is connected with chromatic polynomial*

$$P_G(t) = t^n - c_1 t^{n-1} + c_2 t^{n-2} - \cdots + (-1)^{n-1} c_{n-1} t.$$

Then the sequence $(1, c_1, c_2, \ldots, c_{n-1})$ is log-concave, which means

$$c_{i-1} c_{i+1} \leqslant c_i^2 \quad \textit{for all } i.$$

In particular, it is unimodal, which means

$$1 \leqslant c_1 \leqslant c_2 \leqslant \cdots \leqslant c_{k-1} \leqslant c_k \geqslant c_{k+1} \geqslant \cdots \geqslant c_{n-1}, \quad \textit{for some } k.$$

Proof. This theorem proves a conjecture of Read from 1968. We will not prove the theorem (the proof involves algebraic geometry and singularity theory). □

Exercise 5. 1. Why the name *log-concave*?

2. Prove that a log-concave sequence of positive real numbers is unimodal.

Remark 6. We can prove $1 \leqslant c_1 \leqslant c_2 \leqslant \cdots \leqslant c_{\lfloor \frac{1}{2}(n-1) \rfloor}$.
If G is a tree, then

$$P_G(t) = t(t-1)^{n-1} = \sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^i t^{n-i} \cdot t = t^n - \binom{n-1}{1} t^{n-1} + \binom{n-1}{2} t^{n-2} - \cdots .$$

The sequence $(1, c_1, c_2, \ldots)$ is $(1, \binom{n-1}{1}, \binom{n-1}{2}, \cdots)$, and it is increasing up to the middle term.
Now suppose that G is connected, but not a tree. Then, by definition of a tree, there is an edge $e \in E(G)$ such that $G - e$ is still connected. For $i \leqslant \frac{1}{2}(n-1)$ we notice that

$$P_G(t) = P_{G-e}(t) - P_{G/e}(t) \Longrightarrow c_{i-1}(G) = c_{i-1}(G-e) - (-c_{i-2}(G/e)) = c_{i-1}(G-e) + c_{i-2}(G/e).$$

We know $i \leqslant \frac{1}{2}(n-1)$ and $i - 1 \leqslant \frac{1}{2}(n-2) = \frac{1}{2}(|V(G/e)| - 1)$, hence by induction

$$c_{i-1}(G) \leqslant c_i(G-e) + c_{i-1}(G/e) = c_i(G)$$

which ends the induction step.

Question. What else does the chromatic polynomial count? And how?

Definition 7. *An* orientation *of G is a choice of direction for every edge. This gives a directed graph. If G has m edges, then it has 2^m possible orientations (which might also be isomorphic).*

Definition 8. *An orientation is* acyclic *if it has no closed directed walk. Let $a(G)$ be the number of acyclic orientations of G.*

Theorem 9. (Stanley, 1973) *If G has n vertices, then $a(G) = (-1)^n P_G(-1)$.*

Example 10.

- G is a tree with n vertices, then

$$a(G) = 2^{n-1} = (-1)^n(-1)(-1-1)^{n-1} = (-1)^n P_G(-1),$$

- G is a cycle on n vertices, then

$$a(G) = 2^n - 2,$$
$$(-1)^n P_G(-1) = (-1)^n[(-2)^n + (-1)^n(-2)] = (-1)^n[(-1)^n(2^n - 2)] = a(G),$$

- $G = K_n$, then

$$(-1)^n P_G(-1) = (-1)^n(-1)^{\underline{n}} = (-1)^n(-1)(-1-1)(-1-2)\cdots(-1-(n-1)) = (-1)^n(-1)^n n!.$$

An acyclic orientation is the same as ordering the vertices $v_1, v_2, \ldots, v_n$ (there are $n!$ possibilities to do this) and then choosing the orientation

$$v_i \longrightarrow v_j, \quad \text{whenever } i > j.$$

Proof. Take $e = xy \in E(G)$. Write $a^+(G-e), a^-(G-e), a^0(G-e)$ for the number of acyclic orientations of $G-e$ such that:

- There is a directed walk in $G-e$ from x to y (a^+),
- There is a directed walk in $G-e$ from y to x (a^-),
- There is no directed walk either way (a^0).

Claim. $a(G-e) = a^+(G-e) + a^-(G-e) + a^0(G-e)$.

Proof. An acyclic orientation in $G-e$ cannot have directed walks $x \longrightarrow y$ and $y \longrightarrow x$ at the same time. These three sets are therefore disjoint and they give all the possibilities. □

Claim. $a(G/e) = a^0(G-e)$.

Proof. Take an orientation of $G-e$ with no walk $x \longrightarrow y$ or $y \longrightarrow x$. For any $z \in N_{G-e}(x) \cap N_{G-e}(y)$, the edges xz and yz have the same orientation (if not, there would be a walk $x \longrightarrow z \longrightarrow y$ or $y \longrightarrow z \longrightarrow x$), hence either

$$x \longrightarrow z \text{ and } y \longrightarrow z$$

or

$$z \longrightarrow x \text{ and } z \longrightarrow y.$$

The orientation of $G-e$ determines then an orientation of G/e (the edges xz and yz are compatible under the contraction). This orientation is also acyclic (a directed walk from xy to itself would imply a directed walk in $G-e$ from x or y to y or x). This also works vice versa.
The idea here was that

$$\text{Closed walks in } G/e = \text{Walks } x \longrightarrow y \text{ or } y \longrightarrow x \text{ in } G-e.$$

□

Claim. $a(G) = a^+(G-e) + a^-(G-e) + 2a^0(G-e)$.

Proof. For the first two terms there is only one way to extend the orientation of $G-e$ without closing a cycle in G. In the last case the edge xy can be oriented both ways, since we don't have a walk from x to y or from y to x. □

By these three claims we obtain

$$\begin{aligned} a(G) &= a^+(G-e) + a^-(G-e) + 2a^0(G-e) = \\ &= a^+(G-e) + a^-(G-e) + a^0(G-e) + a^0(G-e) = \\ &= a(G-e) + a^0(G-e) = \\ &= a(G-e) + a(G/e) \end{aligned}$$

We complete the proof by using induction:

- $G = K_1$, then $a(G) = 1 = (-1)^1 P_{K_1}(-1)$,
- Pick an edge $e \in E(G)$, then (by induction assumption)

$$\begin{aligned} a(G) &= a(G-e) + a(G/e) = \\ &= (-1)^n P_{G-e}(-1) + (-1)^{n-1} P_{G/e}(-1) = \\ &= (-1)^n [P_{G-e}(-1) - P_{G/e}(-1)] = \\ &= (-1)^n P_G(-1) \end{aligned}$$

□

Definition 11. *$\alpha \in \mathbb{C}$ is a* chromatic root *if $P_G(\alpha) = 0$ for some graph G.*

Observation 12. *1. Every natural number is a chromatic root,*

2. For any G different from the empty graph, $P_G(0) = 0$,

3. For any G with at least one edge, $P_G(1) = 0$,

4. If α is a chromatic root, then so is $\alpha + 1$,

Proof. We proved in the exercise session that $P_{G+K_1}(\alpha + 1) = (\alpha + 1)P_G(\alpha)$, □

5. The set of chromatic roots is countable (it is a subset of the algebraic numbers).

Proposition 13. *There is no chromatic root in $(-\infty, 0) \cup (0, 1)$.*

Proof. $\alpha < 0$ is not a root of $P_G(t)$, since the coefficients of the polynomial have alternating signs.

Take $\alpha \in (0, 1)$. Because $P_{G \sqcup H}(t) = P_G(t)P_H(t)$, it suffices to prove that $P_G(\alpha) \neq 0$ for any connected graph. Apply the deletion-contraction rule to G, in such a way that all the intermediate graphs are connected. At each step, either G is a tree (and we stop splitting) or there is an edge $e \in E(G)$ such that $G - e$ is still connected.

A branch of this splitting process with i contractions

- ends with an $(n-i)$-vertex tree,
- introduces a sign of $(-1)^i$,
- contributes $t(t-1)^{n-i-1}$ to $P_G(t)$.

Define d_i as the number of branches ending with an $(n-i)$-vertex tree, then

$$P_G(t) = \sum d_i (-1)^i t(t-1)^{n-i-1},$$

and of course we have $d_i \geqslant 0$. Evaluate $P_G(\alpha)$ for $\alpha \in (0, 1)$:

$$sgn\{d_i(-1)^i \alpha(\alpha - 1)^{n-i-1}\} = (-1)^i \cdot 1 \cdot (-1)^{n-i-1} = (-1)^{n-1},$$

which means that all monomials in $P_G(\alpha)$ evaluate to positive or all evaluate to negative, hence $P_G(\alpha) \neq 0$ as $d_i > 0$ for at least one i. □

Remark 14. We used the deletion-contraction principle, but only until we reached trees (since we already know their chromatic polynomial).

Theorem 15. (Jackson, Thomassen) *There are no chromatic roots in* $(-\infty, 0) \cup (0, 1) \cup (1, \frac{32}{27})$*. Moreover, the constant* $\frac{32}{27}$ *is optimal.*

Theorem 16. (Sokal) *The chromatic roots are dense in* $\mathbb{C}$*.*

Theorem 17. (Birkhoff, Lewis) *If* G *is planar, then* $P_G(t) > 0$ *for all* $t \in [5, \infty)$*.*

Remark 18. Moreover, it is conjectured that if G is planar, then $P_G(t) > 0$ for all $t \in [4, \infty)$.