# Graph coloring

**Lecture notes, vol.9**

**Edge Coloring**

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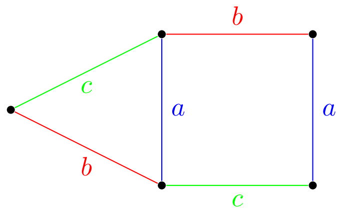
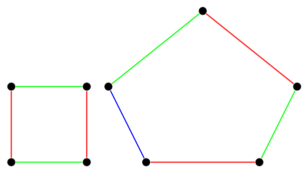
In the next pages, $G$ is always a graph, $V(G)$ its set of vertices and $E(G)$ its set of edges.

**Definition 1.** *An* edge coloring *of $G$ is a function $f : E(G) \to C$ (for some set of colors $C$) such that if $e_1 e_2$ have a common endpoint then $f(e_1) \neq f(e_2)$. The* edge chromatic number *(*chromatic index*) $\chi'(G)$ is the smallest $k$ such that $G$ has an edge-coloring with colors $\{1, \dots, k\}$.*

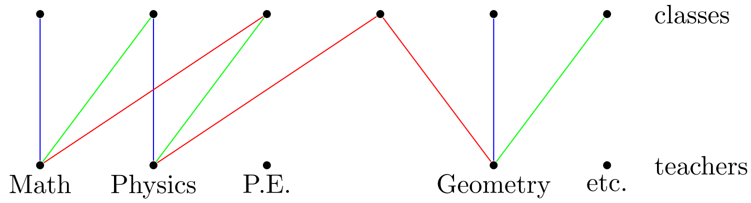



**Example 2.**

- $\chi'(C_n) = \begin{cases} 2 & ,2|n \\ 3 & ,2\nmid n \end{cases}$



- $G$ is a bipartite graph, $V(G) = Classes \cup Teachers$.
  There is an edge $tc$ if $t$ is supposed to teach $c$

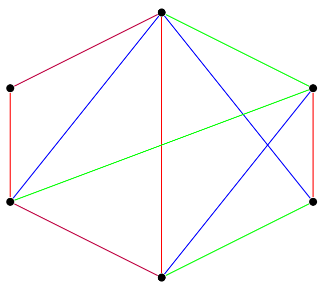



  How many time slots do we need to schedule all lessons?
  Answer: $\chi'(G)$. *Colors* $\equiv$ time slots

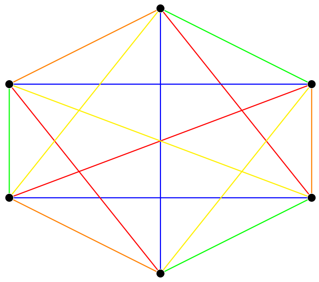
- $\chi'(K_{12}) = 11$

  Season schedule of the Danish soccer league (figures for the case of $K_6$, 6 teams)

  *Proof.*

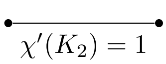


  *colors* $\equiv$ weeks, *color classes* $\equiv$ list of pairs playing that week
  first week second week third week etc.
  (fourth week) is not correct because some team is scheduled two games

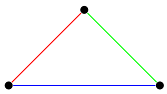


∎

**Lemma 3.** $\chi'(K_n) = \begin{cases} n-1 & ,2|n \\ n & ,2 \nmid n \end{cases}$



$\chi'(K_2) = 1$



$\chi'(K_3) = 3$

*Proof.* $\underline{n = 2k-1}$ Arrange vertices into a regular $n$-gon. Choose a color class consisting of $k$ parallel edges (one vertex is left out). This color class has $n$ possible rotations. These rotations cover all edges of $K_n$. This is an edge-coloring with $n$ colors,so $\chi'(K_n) \leq n$.

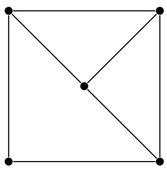
Upper bound: take any edge-coloring of $K_n$. The color class 1 must miss some vertex v (because $2 \nmid n$). Then, $deg(v) = n-1$, and all edges incident to $v$ require $n-1$ colors.Total $\#colors \geq 1+(n-1) = n$

$\underline{n = 2k}$ $deg(v) = n-1$, so we need at least $n-1$ colors, $\chi'(K_n) \geq n-1$. To construct a coloring with $n-1$ colors take the $n-1$ rotations of the following color class: $n-1$ vertices are arranged in a regular $(n-1)$-gon, and one vertex is in the origin. Use $k-1$ parallel edges and one edge from the origin to the vertex on the perimeter being left out. ∎

**Lemma 4.** $\chi'(G) \geq \Delta(G)$

*Proof.* We need at least $\Delta(G)$ colors just to color the edges incident to the vertex of maximum degree. ∎

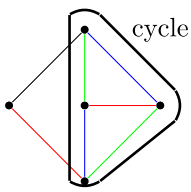
**Example 5.** $\Delta(G) = 3$, but there is no 3-edge coloring (checked in excercises). There is one with 4 colors, so $\chi'(G) = 4$



**Definition 6.** *A* matching *in G is a set of edges, no two of which have a common endpoint [equivalently, 1-regular (every vertex has deg.1) subgraph of G].*

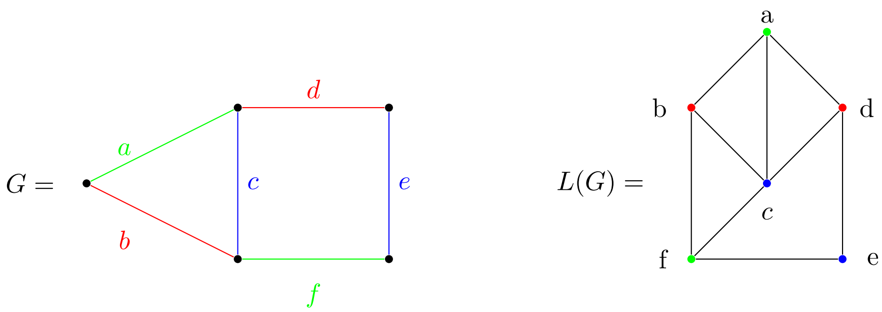
**Observation 7.**

- *If $f$ is an edge-coloring then every color class $f^{-1}(c)$ is a matching.*
- *An edge-coloring with $k$ colors is the same as a partition of $E(G)$ into $k$ matchings*
- *For any two colors $c_1, c_2 \in C$, $c_1 \neq c_2$, $f^{-1}(c_1) \cup f^{-1}(c_2)$ is a disjoint union of cycles and paths.*



*Proof.* The first two parts are clear. For part 3, note that in $f^{-1}(c_1) \cup f^{-1}(c_2)$ every vertex is of degree $\leqslant 2$. ☐

**Definition 8.** *The* line graph *$G$ is the graph $L(G)$ which keeps track of incidences between edges of $G$.*
*Formally: $V(L(G)) = E(G)$*
*$e_1e_2 \in E(L(G))$ if $e_1, e_2$ share a common vertex in $G$ (here $e_i \in E(G), e_i \in V(L(G))$).*

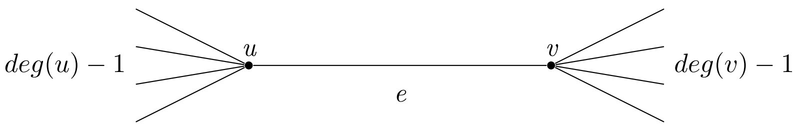



**Example 9.**

<u>**Glossary**</u>

edge in $G \equiv$ vertex of $L(G)$
matching in $G \equiv$ independent set in $L(G)$
edge-coloring of $G \equiv$ vertex coloring of $L(G)$
$\chi'(G)$=$\chi(L(G))$

**Observation 10.** *$\omega(L(G)) \geq \Delta(G)$ because all edges incedent to a fixed vertex $v$, induce a clique in $L(G)$. That implies*
*$\chi'(G) = \chi(L(G)) \geq \omega(L(G)) \geq \Delta(G)$ which we already knew. Moreover:*
$\Delta(L(G)) = \max_{uv \in E(G)}(deg_G(u) + deg_G(v) - 2) \leq 2\Delta(G) - 2$




*Now, greedy coloring gives:*
$\chi'(G) = \chi(L(G)) \leq \Delta(L(G)) + 1 \leq 2\Delta(G) - 1$
*We proved: $\Delta(G) \leq \chi'(G) \leq 2\Delta(G) - 1$*

**Theorem 11.** *(Vizing '64) For any graph $G$ we have $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.*

**Remark 12.** It is NP-hard to recognize if $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$, even for graphs with $\Delta(G) = 3$.
Graphs with $\chi'(G) = \Delta(G)$ are called *Class* 1
Graphs with $\chi'(G) = \Delta(G) + 1$ are called *Class* 2
But we can still indentify some graph classes for which $\chi'(G) = \Delta(G)$.

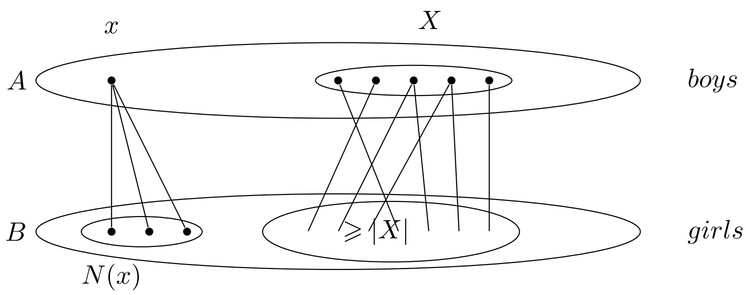
**Theorem 13.** *(König) If $G$ is bipartite then $\chi'(G) = \Delta(G)$*

<u>Hall's marriage theorem</u>(1935-original version): We have a number of boys and girls, each boy fancies some of the girls. Is it possible to arrange marriages, so that each boy marries some girls he likes? Obvious necessary condition: "Each set of $k$ boys ($k \geq 1$) likes, altogether, at least $k$ different girls".

**Theorem 14.** *(Hall) If $G$ is a bipartite graph with parts $V(G) = A \cup B$, such that for every $X \subseteq A$ we have:*

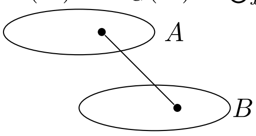
$$|\bigcup_{x \in X} N_G(x)| \geq |X|$$

*then $G$ has a matching of size $|A|$.*



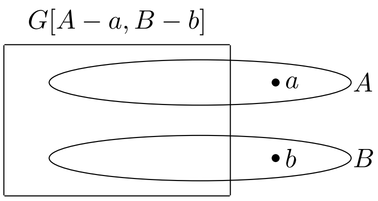

*Proof.* Write $N(X) = N_G(X) = \bigcup_{x\in X} N_G(x)$. Induction on $n = |A|$

(1) If $n = 1$



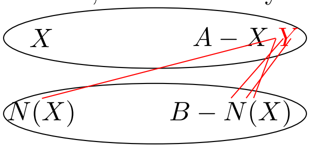

the unique vertex of $A$ has at least one edge.

(2a) For any $X \subsetneq A$, $|N_G(X)| \geq |X| + 1$.
Then take any $a \in A$, $b \in N_G(a)$, match them and use induction on $G[A - a, B - b]$.




Possible because for any $X \subseteq A - a$ we have $|N(X)| \geq |X| + 1 - 1 = |X|$

(2b) $N_G(X) = |X|$ for some $X \subsetneq A$.
then: by induction find a matching in $G[X, N(X)]$
Also, there is a matching in $G[A - X, B - N(X)]$
Possible,because for any $Y \subseteq A - X$




$|Y| + |X| = |Y \cup X| \leqslant |N_G(Y \cup X)| = |N(X)| + |N_G(Y) \cap (B - N(X))| = |X| + |N_G(Y) \cap (B - N(X))|$
So $|Y| \leqslant |N_G(Y) \cap (B - N(X))|$ and therefore induction applies to $G[A - X, B - N(X)]$ □