Root and *root finding*
are concepts familiar to most branches of mathematics.

Square roots and higher roots of graphs

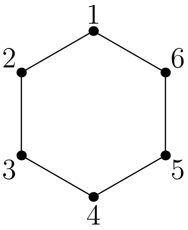
Anna Adamaszek, Michał Adamaszek

DIMAP, University of Warwick, UK

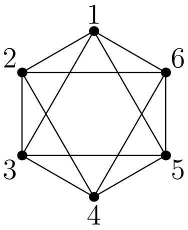
Powers and roots of graphs

Definition

If H is a graph, its *r-th power* $G = H^r$ is the graph on the same vertex set such that two distinct vertices are adjacent in G if their distance in H is at most r. We call H the *r-th root* of G.



H



H^2

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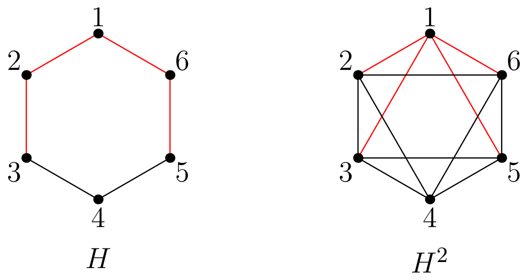

Problems related to graph roots

- Does G have an r-th root?
- How to compute the r-th root?
- Is the r-th root of G unique?

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Theorem (Motwani, Sudan '94, Le, Nguyen '09)

It is NP-complete to decide if G has a square/r-th root.

Tree roots

Lin,Skiena '95, Kearney,Corneil '98, Chang,Ko,Lu '06

r-th tree roots can be found in linear time.

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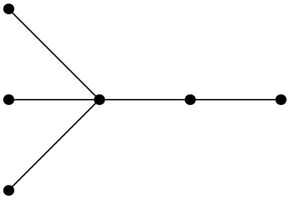
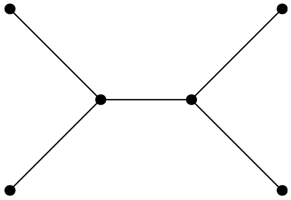
Warning! Very non-unique for $r \geq 3$!





are both 3-rd roots of the complete graph!

Large-girth roots

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The *girth* of a graph is the length of its shortest cycle.

Farzad,Lau,Le,Tuy '09

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AA,MA '09

- r-th roots of girth at least $2r + 3$ can be found in poly-time.
- Square roots of girth at least $r + 2$ are NP-hard to find.

How it works

Task. We know $B_x = B_r(x)$ for for each vertex $x \in H$. What can we say about H?
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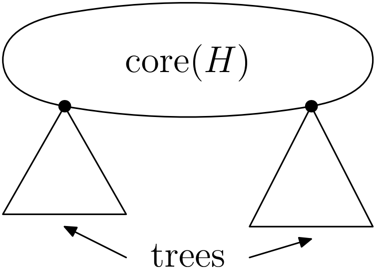

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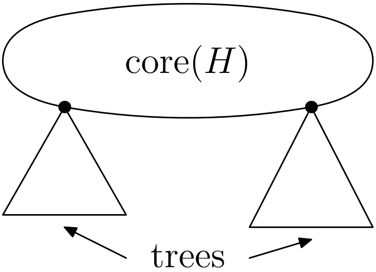

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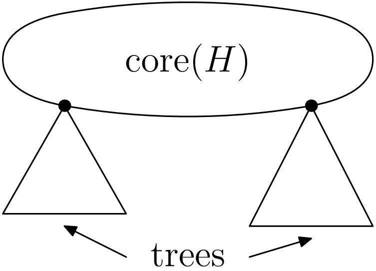

- Miracle 2. We can compute *all* r-th roots of the core, and there is linearly many of them.
- Miracle 3. Given each core we can attach the trees in the right places.

Roots with no leaves

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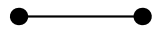

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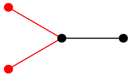

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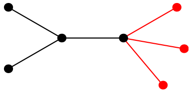

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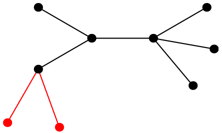
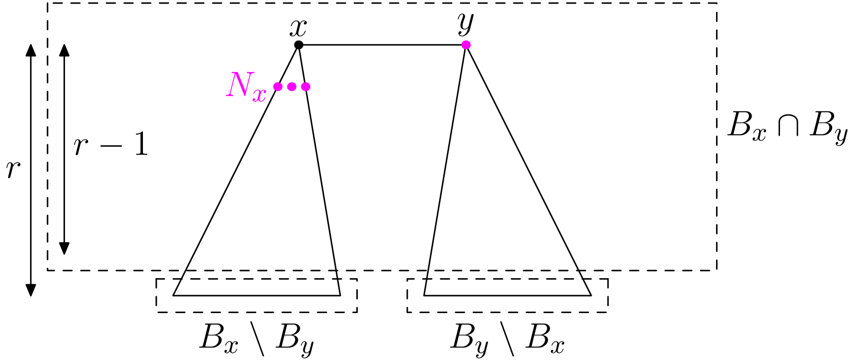
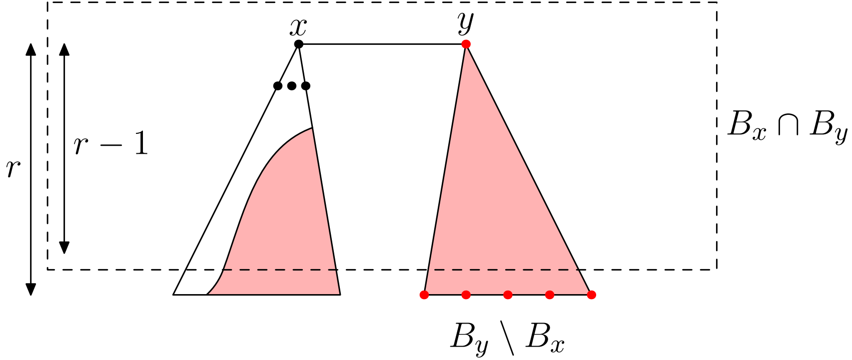


The submiracle explained



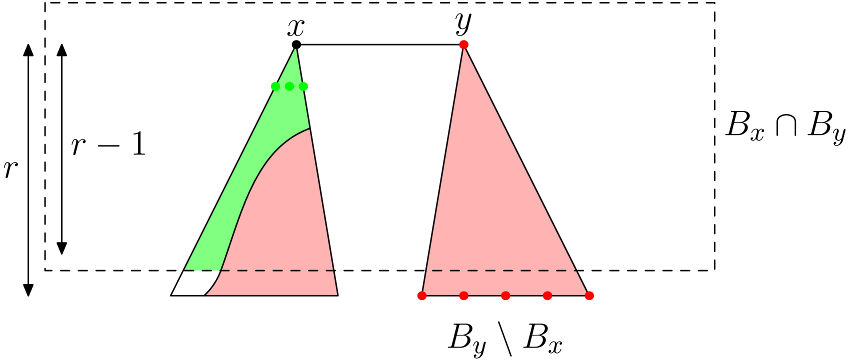
- $B_x \cup B_y$ is a tree
- the leaves are $B_x \setminus B_y$ and $B_y \setminus B_x$
- goal: find N_x

The submiracle explained



- The set $\mathcal{R} = \bigcup_{v \in B_y \setminus B_x} B_v$ is marked in red

The submiracle explained



- $\mathcal{R} = \bigcup_{v \in B_y \setminus B_x} B_v$
- $\mathcal{G} = B_x \cap B_y \setminus \mathcal{R} \setminus \{x\}$
- $N_x \subseteq \mathcal{G}$

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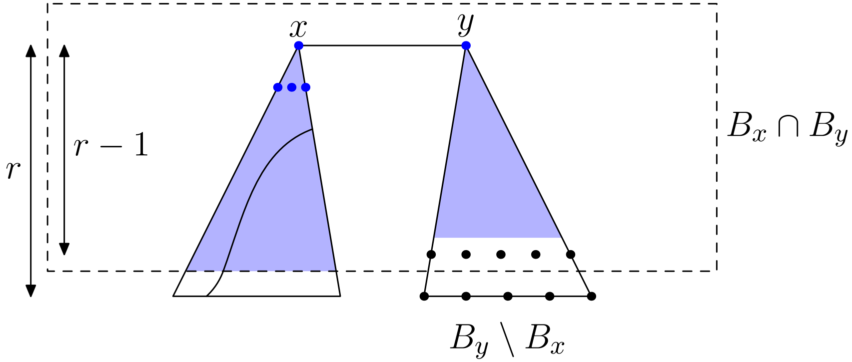

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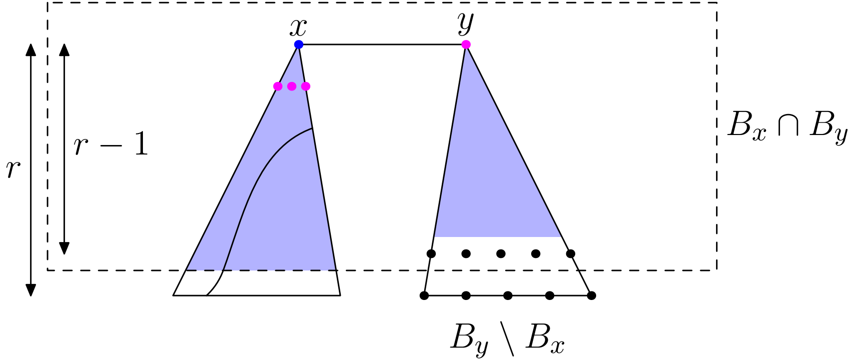

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- $N_x = B_x \cap B_y \cap \bigcap_{v \in \mathcal{B}} B_v \setminus \{x\}$ — neighbours of x

If we had an algorithm...

Input: G

Output:

$$\text{a graph } H\text{, of girth at least 5, such that } H^2 = G$$

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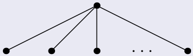
- 

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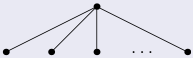
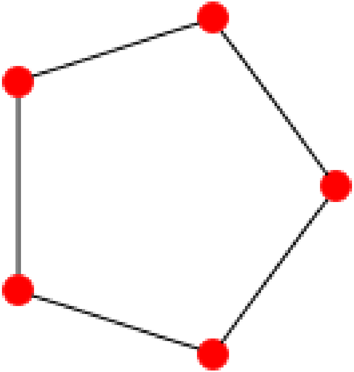
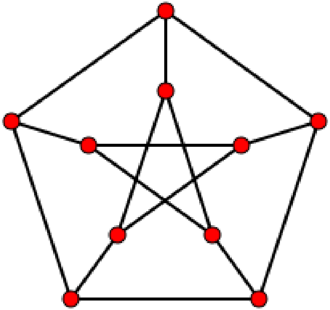
- 
- a graph H of diameter 2 and girth 5

Cycle C_5



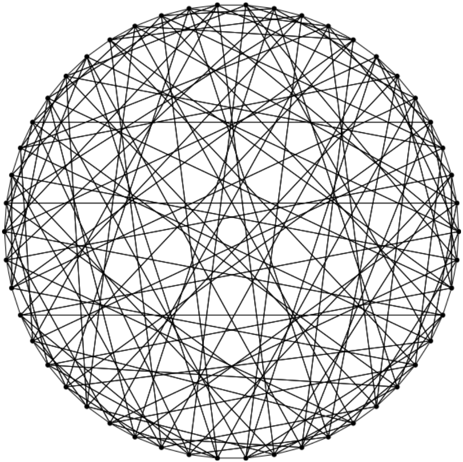
$|V| = 5 = 2^2 + 1$, 2-regular, girth=5, diam=2

Petersen graph



$|V| = 10 = 3^2 + 1$, 3-regular, girth=5, diam=2

Hoffman-Singleton graph



$|V| = 50 = 7^2 + 1$, 7-regular, girth=5, diam=2

A mysterious $(57, 5)$-cage

$|V| = 3250 = 57^2 + 1$, 57-regular, girth=5, diam=2

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Input: K_{3250}

Output:

The mysterious $(57, 5)$-cage (a 57-regular graph with diam=2, girth=5 and 3250 vertices).

Uniqueness of square roots

Ross,Harary '60

The tree square root of a graph is unique up to iso.

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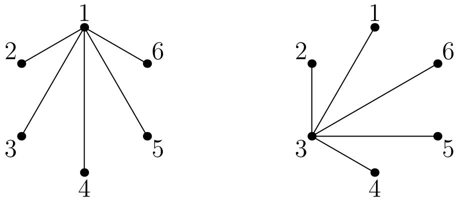

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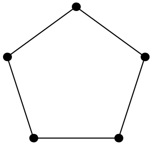
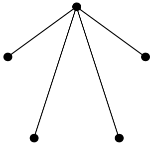
Stop!

The square root of girth ≥ 5 of a graph may not be unique.





Uniqueness of square roots with no leaves

Levenshtein et.al. '08

The square root of girth ≥ 7 and no leaves of a graph is unique.

Uniqueness of square roots with no leaves

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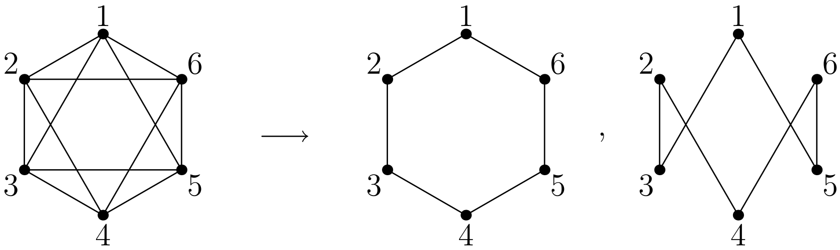
Stop!

False for girth ≥ 6.



OK, what about higher roots?

CONJECTURE, Levenshtein '08

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The r-th root of girth $\geq 2.5r$ and no leaves of a graph is unique.

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AA,MA '09; Miracle 2

Each G has at most $O(|G|)$ r-th roots of girth $\geq 2r+3$ and no leaves.

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